12. Find the equilibrium vector for the transition matrix

\[
P = \begin{bmatrix}
.5 & .2 & .3 \\
.1 & .4 & .5 \\
.2 & .2 & .6
\end{bmatrix}
\]

We need to find the probability vector \( V \) such that \( VP = V \). To solve this system set the vectors equal to zero and factor \( V \).

\[
VP = V \\
VP - V = 0 \\
V(P-I) = 0,
\]

where \( I \) is the identity matrix. Recall that probability vectors are row vectors. Since \( P \) is \( 3 \times 3 \) the equilibrium vector must be of the form \( V = [v_1, v_2, v_3] \) and \( I \) is the \( 3 \times 3 \) identity.

First calculate \( P - I \) and then multiply (on the left) by \( V \).

\[
P - I = \begin{bmatrix}
-.5 & .2 & .3 \\
.1 & -.6 & .5 \\
.2 & .2 & -.4
\end{bmatrix}
\]

\[
V(P-I) = [v_1, v_2, v_3] \begin{bmatrix}
-.5 & .2 & .3 \\
.1 & -.6 & .5 \\
.2 & .2 & -.4
\end{bmatrix} = [0, 0, 0]
\]

Multiplying yields

\[
-.5v_1 + .1v_2 + .2v_3 = 0 \\
.2v_1 - .6v_2 + .2v_3 = 0 \\
.3v_1 + .5v_2 - .4v_3 = 0
\]

Here we could try to solve the \( 3 \times 3 \) system using the methods from Chapter 2. The only problem is the solution will have a free variable. (Use your calculator to verify this for yourself.) To find the solution we must add another equation. Since \( V \) is a probability vector the sum of the entries must be 1, \( v_1 + v_2 + v_3 = 1 \). Rewriting all of the equations as a matrix product yields

\[
\begin{bmatrix}
-.5 & .1 & .2 \\
.2 & -.6 & .2 \\
.3 & .5 & -.4
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \begin{bmatrix}0 \\0 \\1
\end{bmatrix}
\]

Lastly write the system as an augmented matrix and solve with your calculator using \texttt{rref}.

\[
\texttt{rref}\left( \begin{bmatrix}
-.5 & .1 & .2 & 0 \\
.2 & -.6 & .2 & 0 \\
.3 & .5 & -.4 & 0 \\
1 & 1 & 1 & 0
\end{bmatrix} \right) = \begin{bmatrix}1 & 0 & 0 & .25 \\0 & 1 & 0 & .25 \\0 & 0 & 1 & .5 \\0 & 0 & 0 & 0
\end{bmatrix}
\]

Thus the equilibrium vector is \( V = [.25, .25, .5] \).