Sample Exam — Linear Equations, Matrices and Linear Programming

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Begin Exam.

Directions: In each of the following exercises, circle the “best” answer on the exam. Each problem is worth 5 points.

Use the following matrices for questions 1-3.

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 1 & -1 \end{bmatrix} \]

1. Compute \(-2A\).
   (a) \[ \begin{bmatrix} -2a & -2b \\ -2c & -2d \end{bmatrix} \]
   (b) \[ \begin{bmatrix} -2a & -2b \\ c & d \end{bmatrix} \]
   (c) \[ \begin{bmatrix} -2a & b \\ -2c & d \end{bmatrix} \]
   (d) \[ \begin{bmatrix} -4 & -2 \\ -6 & 10 \end{bmatrix} \]
   (e) Not possible.

2. Compute \(A + 2B\).
   (a) \[ \begin{bmatrix} a - 3 & b + 1 \\ c + 2 & d - 2 \end{bmatrix} \]
   (b) \[ \begin{bmatrix} a - 6 & b + 2 \\ c + 4 & d - 4 \end{bmatrix} \]
   (c) \[ \begin{bmatrix} -6 & 2 \\ 4 & -4 \end{bmatrix} \]
   (d) \[ \begin{bmatrix} a & b - 6 & 2 \\ c & d & 4 & -4 \end{bmatrix} \]
   (e) Not possible.

3. Compute \(B - 2C\).
   (a) \[ \begin{bmatrix} -2 & -5 \\ -5 & -15 \\ -2 & 2 \end{bmatrix} \]
   (b) \[ \begin{bmatrix} -2 & -5 \\ -5 & -15 \end{bmatrix} \]
   (c) \[ \begin{bmatrix} -4 & -6 \\ -6 & -9 \\ 1 & -3 \end{bmatrix} \]
   (d) \[ \begin{bmatrix} -6 & -9 \\ 1 & -3 \end{bmatrix} \]
   (e) Not possible.

4. The result of performing the row operation \(-2r_1 + r_3\) to replace \(r_3\) given the augmented matrix
   \[ \begin{bmatrix} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ 2 & 5 & 4 & 6 \end{bmatrix} \]
   is
   (a) \[ \begin{bmatrix} 1 & -3 & -5 & -2 \\ 0 & 1 & 6 & 9 \\ 2 & 5 & 4 & 6 \end{bmatrix} \]
   (b) \[ \begin{bmatrix} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ 0 & 11 & 14 & 10 \end{bmatrix} \]
   (c) \[ \begin{bmatrix} -2 & 6 & 10 & 4 \\ 2 & -5 & -4 & 5 \\ 0 & 5 & 4 & 6 \end{bmatrix} \]
   (d) \[ \begin{bmatrix} 1 & -3 & -5 & -2 \\ 0 & 1 & 6 & 9 \\ 0 & 11 & 14 & 10 \end{bmatrix} \]
   (e) None of these
**Directions:** Place the solution to each of the following exercises on your own paper. For full credit all work must be shown.

**Exercise 1. (10 points)** Grandfather clocks use pendulums to keep accurate time. The relationship between the length of a pendulum $L$ and the time $T$ for one complete oscillation can be determined from the data in the table

<table>
<thead>
<tr>
<th>$L$ (ft)</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$ (sec)</td>
<td>1.11</td>
<td>1.36</td>
<td>1.57</td>
<td>1.76</td>
<td>1.92</td>
<td>2.08</td>
<td>2.22</td>
</tr>
</tbody>
</table>

(a) Plot the data on your graph paper. Put length $L$ on the horizontal axis and time $T$ on the vertical axis. Clearly scale the axes.

(b) Find the equation of the regression line. Round the constants to 4 decimal places. Draw the regression line on your graph.

(c) Predict the time for one period of a pendulum that is 5 feet long. Round your answer to 2 decimal places. On your graph indicate the point with its coordinates.

(d) Explain why it is acceptable to use the equation to make the prediction above.

**Exercise 2. (10 points)** Given the system of equations

\[
\begin{align*}
x + y &= -1 \\
2x - y &= 10
\end{align*}
\]

Express the system as an augmented matrix. Use row operations to write the matrix in reduced row echelon form (Gauss-Jordan method). Record the solution to the system.

**Exercise 3.**

(a) (1 point each) Given below is a list of matrices and their corresponding dimension.

<table>
<thead>
<tr>
<th>Name</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimension</td>
<td>$2 \times 3$</td>
<td>$2 \times 3$</td>
<td>$3 \times 3$</td>
<td>$3 \times 1$</td>
<td>$1 \times 2$</td>
</tr>
</tbody>
</table>

If it is possible, state the dimension of the matrix resulting from the arithmetic operations. If a resultant matrix is not possible, give a reason.

a. $A + B$

b. $A \ast C \ast D$

c. $(A - B) \ast C$

d. $C \ast (A - B)$

e. $C^{-1} \ast D$

f. $E \ast A \ast D$
(b) (4 points) If 
\[
F = \begin{bmatrix}
1 & 2 & 4 \\
-1 & 3 & 1 \\
0 & -2 & 5 \\
\end{bmatrix}
\quad \text{and} \quad
G = \begin{bmatrix}
1 & 9 \\
8 & 7 \\
-1 & 3 \\
\end{bmatrix}
\]
show how you would get the number in the third row and second column of \(FG\).

EXERCISE 4. (10 points each) Solve each of the systems of linear equations below by any method. Clearly indicate the method used to reach your solution. If no solution exists, explain your conclusion. If there are infinitely many solutions express your answer in terms of the free variable(s) and give two particular solutions.

(a) 
\[
\begin{align*}
x - y - z &= 9 \\
x + y + 2z &= -9 \\
-2x + 2y + 2z &= -18
\end{align*}
\]

(b) 
\[
\begin{align*}
x - y - z &= 9 \\
x + y + 2z &= -9 \\
-2x + 2y + 2z &= -9
\end{align*}
\]

EXERCISE 5. (10 points) Melody has $45,000 to invest and wishes to receive an annual income of $4,290 from this money. She has chosen investments that pay 5%, 8% and 12% simple interest. How much should she invest at each rate if she wants the interest from the 8% account to be twice the interest from the 5% account?

(a) Clearly identify your variables and set up a system of equations.

(b) Use any method to find the amount invested at each rate. Record your results and indicate the method used.

EXERCISE 6. (10 points) A town has 3 industries: coal, electric and railroad. The input-output matrix \(A\) is 
\[
A = \begin{bmatrix}
0.00 & 0.60 & 0.40 \\
0.15 & 0.05 & 0.10 \\
0.30 & 0.10 & 0.00
\end{bmatrix}
\]
with demand matrix 
\[
D = \begin{bmatrix}
$1,500,000 \\
$1,000,000 \\
$700,000
\end{bmatrix}
\]. Find the production matrix \(X\) necessary to support this demand. Round your solution to the nearest whole number.

EXERCISE 7. (10 points) Maximize the objective function 
\[z = x + y\] subject to the constraints
\[
\begin{align*}
7x + 6y & \leq 42 \\
x + 2y & \leq 10 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

(a) On your graph paper plot the constraints and clearly indicate the feasible region.

(b) Show your work to find all corner points. Clearly label all corner points on your graph.

(c) Solve the linear programming problem.
Solutions to Quizzes

Solution to Question 1: (a) \(-2A = -2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2a & -2b \\ -2c & -2d \end{bmatrix}\)

Solution to Question 2: (b) \(A + 2B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + 2 \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} a - 6 & b + 2 \\ c + 4 & d - 4 \end{bmatrix}\)

Solution to Question 3: (e) The dimension of \(B\) is \(2 \times 2\) and the dimension of \(C\) is \(3 \times 2\). Since the number of columns of \(B\) do not equal the number of rows in \(C\) it is not possible to compute \(B - 2C\).

Solution to Question 4: (b) Given the matrix \(\begin{bmatrix} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ 2 & 5 & 4 & 6 \end{bmatrix}\) the result of \(-2r_1 + r_3\) replacing \(r_3\) is \(\begin{bmatrix} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ 0 & 11 & 14 & 10 \end{bmatrix}\)
Solutions to Exercises

Exercise 1(a) The data should look similar to the graph below.

Exercise 1(b) The equation of the regression line is $T = .3800L + .7386$ where the constants are rounded to 4 decimal places.

Exercise 1(c) For $L = 5$ the period of the pendulum is $T = .3800(5) + .7386 = 2.6386$. Rounding the answer to 2 decimal places $T = 2.64$ seconds.
Exercise 1(d) The correlation coefficient is $r = 0.9865801048$. Since the correlation coefficient is close to 1, there is relatively good linear fit to the data. The good fit allows one to make predictions with some confidence in the results.

Exercise 2. Given the system of equations
\[
\begin{align*}
    x + y &= -1 \\
    2x - y &= 10
\end{align*}
\]
the augmented matrix is
\[
\begin{bmatrix}
1 & 1 & -1 \\
2 & -1 & 10
\end{bmatrix}.
\]
With the pivot of 1 in the upper left corner get zero below the pivot
\[
\begin{bmatrix}
1 & 1 & -1 \\
0 & -3 & 12
\end{bmatrix}, -2R1 + R2

\begin{bmatrix}
1 & 1 & -1 \\
0 & 1 & -4
\end{bmatrix}, (-1/3)R2
\]
Finish by using the pivot in the second row to get zero above the pivot.
\[
\begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & -4
\end{bmatrix}, -R2 + R1
\]
The solution to the system is $x = 3$ and $y = -4$.

Exercise 3(a)
a. $A + B$ is $2 \times 3$

b. $A \cdot C \cdot D$ is $2 \times 1$.

c. $(A - B) \cdot C$ is $2 \times 3$.

d. $C \cdot (A - B)$, Dimension Mismatch. It is not possible to multiply $(3 \times 3) \cdot (2 \times 3)$.

e. $C^{-1} \cdot D$ is $3 \times 1$.

f. $E \cdot A \cdot D$ is $1 \times 1$.

□

**Exercise 3(b)** To find the entry in the third row and second column of $FG$ multiply the third row of $F$ by the second column of $G$. To compute the row times column multiply the corresponding entries in the row and column and add the results. $[FG]_{32} = (0)(9) + (-2)(7) + (5)(3) = 0 - 14 + 15 = 1$. □

**Exercise 4(a)** The augmented matrix for

\[
\begin{align*}
x - y - z &= 9 \\
x + y + 2z &= -9 \\
-2x + 2y + 2z &= -18
\end{align*}
\]

is

\[
\begin{bmatrix}
1 & -1 & -1 & 9 \\
1 & 1 & 2 & -9 \\
-2 & 2 & 2 & -18
\end{bmatrix}
\]

Use the calculator to compute the reduced row-echelon form:

\[
\text{rref} \begin{bmatrix}
1 & -1 & -1 & 9 \\
1 & 1 & 2 & -9 \\
-2 & 2 & 2 & -18
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 1/2 & 0 \\
0 & 1 & 3/2 & -9 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The matrix has pivots in the $x$ and $y$ columns with $z$ the free variable. Solving for $x$ and $y$ yields

\[
x = \frac{-1}{2}z \\
y = -9 - \frac{3}{2}z \\
z = \text{free}
\]

All solutions are of the form $(-\frac{1}{2}z, -9 - \frac{3}{2}z, z)$. To find particular solutions choose specific values for $z$. At $z = 0$ the solution is $(0, -9, 0)$. At $z = 2$ the solution is $(-1, -12, 2)$. □

**Exercise 4(b)** The augmented matrix for

\[
\begin{align*}
x - y - z &= 9 \\
x + y + 2z &= -9 \\
-2x + 2y + 2z &= -9
\end{align*}
\]

is

\[
\begin{bmatrix}
1 & -1 & -1 & 9 \\
1 & 1 & 2 & -9 \\
-2 & 2 & 2 & -9
\end{bmatrix}
\]

Use the calculator to compute the reduced row-echelon form:
The last row results in the falsity $0 = 1$. Hence there is no solution.  

**Exercise 5(a)** Let

\[
\begin{align*}
&x = \text{amount invested at 5\% interest} \\
y = \text{amount invested at 8\% interest} \\
z = \text{amount invested at 12\% interest}
\end{align*}
\]

The total amount invested translates to $x + y + z = 45,000$. The total interest earned is $0.05x + 0.08y + 0.12z = 4290$. The interest from the 8\% is twice the interest from the 5\% translates to $0.08y = 2(0.05x)$. This results in a system of 3 equations with 3 unknowns.

\[
\begin{align*}
x + y + z &= 45000 \\
0.05x + 0.08y + 0.12z &= 4290 \\
-0.1x + 0.08y + 0z &= 0
\end{align*}
\]

**Exercise 5(b)** The augmented matrix for the system is

\[
\begin{bmatrix}
1 & 1 & 1 & 45000 \\
0.05 & 0.08 & 0.12 & 4290 \\
-0.1 & 0.08 & 0 & 0
\end{bmatrix}
\]

Use the calculator to compute the reduced row-echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 & 9250 \\
0 & 1 & 0 & 11562.50 \\
0 & 0 & 1 & 24187.50
\end{bmatrix}
\]

$9250$ is invested in the 5\% account, $11,562.50$ is invested in the 8\% account and $24,187.50$ is invested in the 12\% account.  

**Exercise 6.** For the open input-output model one must solve $X - AX = D$. There are two methods to solve the system:

1. **Inverse Matrices**

\[
X = (I - A)^{-1}D
\]

2. **Reduced Row-Echelon Form**

\[
\text{rref} ([I - A], D)
\]

Given $A = \begin{bmatrix} 0.00 & 0.60 & 0.40 \\ 0.15 & 0.05 & 0.10 \\ 0.30 & 0.10 & 0.00 \end{bmatrix}$ and $D = \begin{bmatrix} 1,500,000 \\ 1,000,000 \\ 700,000 \end{bmatrix}$ the augmented matrix is

\[
\begin{bmatrix}
1 & -0.60 & -0.40 & 1500000 \\
-0.15 & 0.95 & -0.10 & 1000000 \\
-0.30 & -0.10 & 1 & 700000
\end{bmatrix}
\]
\[
\text{rref}[(I - A), D] = \begin{bmatrix}
1 & 0 & 0 & 3311797.753 \\
0 & 1 & 0 & 1772471.91 \\
0 & 0 & 1 & 1870786.517
\end{bmatrix}
\]

Rounding solutions to the nearest whole number results in $3,311,798$ in coal, $1,772,472$ in electricity, and $1,870,787$ in railroad needed to satisfy the demand.

Exercise 6

Exercise 7(a) Given the system of linear equations

\[
\begin{align*}
7x + 6y &\leq 42 \\
x + 2y &\leq 10
\end{align*}
\]

\[
x \geq 0 \\
y \geq 0
\]

Use a table to organize the information needed to plot the feasible region.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>x-intercept, ( y = 0 )</th>
<th>y-intercept, ( x = 0 )</th>
<th>Test ((0,0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 7x + 6y \leq 42 )</td>
<td>((6,0))</td>
<td>((0,7))</td>
<td>(0 \leq 42, \text{ True})</td>
</tr>
<tr>
<td>( x + 2y \leq 10 )</td>
<td>((10,0))</td>
<td>((0,5))</td>
<td>(0 \leq 10, \text{ True})</td>
</tr>
</tbody>
</table>

\[
\text{rref} \begin{bmatrix}
7 & 6 & 42 \\
1 & 2 & 10
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 3 \\
0 & 1 & 7/2
\end{bmatrix}
\]

The last corner point is \((3, 7/2)\).

The final graph should look similar to the one below.
Exercise 7(c) To solve the linear programming problem check the objective functions at the corner points

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Objective $z = x + y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>$z = 0$</td>
</tr>
<tr>
<td>(6, 0)</td>
<td>$z = 6$</td>
</tr>
<tr>
<td>(0, 5)</td>
<td>$z = 5$</td>
</tr>
<tr>
<td>$\left(3, \frac{7}{2}\right)$</td>
<td>$z = \frac{13}{2}$, MAX</td>
</tr>
</tbody>
</table>

The objective function has a maximum of $z = 13/2$ at the point $(3, 7/2)$. 