Sets

Probability Definitions

Conditional Probability

Independent Events

Bayes’ Theorem

Math 117

Chapter 7 — Sets and Probability

Flathead Valley Community College
1. Sets

A set is a well-defined collection of specific objects.

Each item in the set is called an element or a member. Curly braces, \{ \}, are used to list the elements in a set. The \( \in \) symbol is used to denote that an element is in a given set.

For example consider the set \( A = \{2, 5, 6, 7, 9, 11\} \).

\[ 5 \in A \quad \text{and} \quad 8 \notin A \]

Sets may also be written using set builder notation of the form

\[ \{x \mid x \text{ has property } P\} \]

For example the set of even integers can be written

\[ E = \{x \mid x = 2k \text{ for } k \text{ any integer }\} \]

The empty set, denoted \( \emptyset \), is the set that contains no elements. Do not but braces around the empty set. \( \{\emptyset\} \) actually represents the set that contains the empty set.

The universal set, denoted \( U \), is the set that includes all the objects being discussed.
The **number** of elements in a set $B$ is denoted $n(B)$.

For example $n(A) = 6$ and $n(\emptyset) = 0$.

Two sets are **equal** if they contain exactly the same elements.

For example $\{1, 2, 3\} = \{2, 3, 1\}$, but $\{1, 2, 3\} \neq \{1, 2, 4\}$

### 1.1. Subsets

Set $A$ is a **subset** of set $B$, denoted $A \subseteq B$, if every element of $A$ is also an element of $B$. Set $A$ is a **proper subset** of set $B$, denoted $A \subset B$, if $A \subseteq B$ and $A \neq B$.

With all of this notation it is often nice to have a diagram representing the sets. The diagram for sets is called a **Venn diagram**. The rectangular box represents the universal set $U$. The interior of each circle represents all of the elements contained in a set. Here is a venn diagram that shows that $A$ is a subset of $B$. 

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Properties of Subsets

1. For any set $A$,
   \[ \emptyset \subseteq A \quad \text{and} \quad A \subseteq A. \]

2. A set of $n$ distinct elements has $2^n$ subsets.
1.2. Set Operations

Let $A$ be any set, with $U$ representing the universal set. Then the complement of $A$, denoted $A'$, is

$$A' = \{x \mid x \notin A \text{ and } x \in U\}.$$
The **intersection** of sets $A$ and $B$, is

$$A \cap B = \{x|x \in A \text{ and } x \in B\}.$$
$A$ and $B$ are said to be **disjoint** when $A \cap B = \emptyset$.

![Disjoint Sets](image)

Figure 4: Disjoint Sets
The **union** of sets $A$ and $B$, is

$$A \cup B = \{x | x \in A \text{ or } x \in B\}.$$  

![Venn Diagram](U.png)

**Figure 5: $A \cup B$**

### Counting with Set Operations

1. **Union Rule for Sets**
   
   For all sets $A$ and $B$,
   
   $$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$
2. Union Rule for Disjoint Sets

If \( A \) and \( B \) are disjoint sets,

\[
n(A \cup B) = n(A) + n(B).
\]
2. Probability Definitions

In probability, an experiment is any activity that has observable results. Each of the possible results is called an outcome. The set of all possible outcomes is called the sample space.

An event is a subset of the sample. An event with only one possible outcome is called a simple event. If event $E = \emptyset$, then $E$ is called an impossible event. If event $E$ equals the entire sample space $S$, then $E$ is called a certain event.

Two events $E$ and $F$ are mutually exclusive events if $E \cap F = \emptyset$.

2.1. Basic Probability Principle

Let $S$ be a sample space of equally likely outcomes, and let event $E$ be a subset of $S$. Then the probability that event $E$ occurs is

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ is the number of elements in the event $E$ and $n(S)$ is the number of elements in the sample space.
2.2. Probability Properties

1. For any event $E$, $0 \leq P(E) \leq 1$.

2. **Union Rule for Probability**
   For any events $E$ and $F$ from a sample space $S$,
   
   $$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

3. **Union Rule for Mutually Exclusive Events**
   For mutually exclusive events $E$ and $F$,
   
   $$P(E \cup F) = P(E) + P(F).$$

4. **Complement Rule**
   
   $$P(E) = 1 - P(E') \quad \text{and} \quad P(E') = 1 - P(E)$$

2.3. Odds

The **odds in favor** of an event $E$ are defined as the ratio of $P(E)$ to $P(E')$, or

$$\frac{P(E)}{P(E')} , P(E') \neq 0.$$ 

If the odds favoring event $E$ are $m$ to $n$, then

$$P(E) = \frac{m}{m+n} \quad \text{and} \quad P(E') = \frac{n}{m+n}$$
2.4. More Probability Properties

Let $S$ be a sample space consisting of $n$ distinct outcomes, $s_1, s_2, \ldots, s_n$. An acceptable probability assignment consists of assigning to each outcome $s_i$ a number $p_i$ (the probability of $s_i$) according to these rules.

1. The probability of each outcome is a number between 0 and 1.

$$0 \leq p_1 \leq 1, \ 0 \leq p_2 \leq 1, \ \ldots \ 0 \leq p_n \leq 1.$$ 

2. The sum of the probabilities of all possible outcomes is 1.

$$p_1 + p_2 + \cdots + p_n = 1.$$ 

A sample space that satisfies the previous properties is called a probability distribution.
3. Conditional Probability

The probability of an event $E$ given that event $F$ occurs is called *conditional probability*.

The **conditional probability** of event $E$ given event $F$, written $P(E|F)$, is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, \text{ where } P(F) \neq 0.$$  

The conditional probability can also be written as a product in order to find the probability of the intersection of two sets.

3.1. Product Rule of Probability

If $E$ and $F$ are events, then $P(E \cap F)$ may be found by either of the formulas.

$$P(E \cap F) = P(F) \cdot P(E|F) \quad \text{or} \quad P(E \cap F) = P(E) \cdot P(F|E)$$
4. **Independent Events**

Events $E$ and $F$ are **independent events** if

$$P(F|E) = P(F) \quad \text{or} \quad P(E|F) = P(E).$$

4.1. **Product Rule for Independent Events**

Events $E$ and $F$ are independent events if and only if

$$P(E \cap F) = P(E) \cdot P(F).$$
5. Bayes’ Theorem

5.1. Special Case

\[ P(F|E) = \frac{P(F) \cdot P(E|F)}{P(F) \cdot P(E|F) + P(F') \cdot P(E|F')} \]

5.2. General Case

\[ P(F_i|E) = \frac{P(F_i) \cdot P(E|F_i)}{P(F_1) \cdot P(E|F_1) + P(F_2) \cdot P(E|F_2) + \cdots + P(F_n) \cdot P(E|F_n)} \]

Using Bayes’ Theorem

1. Create a tree diagram with branches \( F_1, F_2, \ldots, F_n \) labeling each branch with its probability.

2. From the end of each branch, draw a branch for event \( E \). Label this branch with the probability \( P(E|F_i) \).

3. Multiply the probabilities along each branch giving \( P(F_i) \cdot P(E|F_i) = P(F_i \cap E) \).

4. To find \( P(F_i|E) \) divide the probability of the branch for \( F_i \) by the sum of the probabilities of all of the other branches producing \( E \).