Applications of Matrices to Civil Engineering

Will Hay

May 12, 2010
Abstract

The goal of this paper is to show the applications of linear algebra to civil engineering; with emphasis on linear equations and their independence/dependence.
1 What is Linear Algebra?

Linear Algebra is the study of vectors (as either vectors or linear equations). Mostly, these vectors are found in matrices, which is an easy way to compress the information given by a series of equations or vectors. The applications of linear algebra are used heavily in many types of engineering, but this document will focus primarily on the applications to Civil engineering.

In these fields, the use of linear algebra is common, especially with problems such as trusses, beams, supports, material mechanics, fluid dynamics, and mechanical stresses. With these sorts of problems, there are large numbers of variables for us to solve. Later on we will solve a truss problem, using the principles we have explained.

2 Statics

Statics is the study of problems at equilibrium, meaning that the goal is to use the fact that it is stationary to find the force exerted on the body. Statics is usually the first real engineering class taken by most students. Along with statics goes Mechanics of Materials, but we will not need to use that in the cases we will see here.

2.1 Forces

To first look at a problem, it must be determined whether the problem is at a static equilibrium. This is as simple as looking at whether its moving at a constant velocity/no velocity. From this, we can make assumptions which in turn let us solve the problems. In static equilibrium, we can assume that the forces are balanced with each other, resulting in a net acceleration of zero. From this, we can say that the forces in the direction of the $x$–axis are equal to zero, as are those in the $y$– and $z$–axis. In addition to that, we can use torque, called a ”moment”, about any axis or point. And using these equations we can solve for stresses, strains, torques and shear forces exerted upon the bodies.
3 Torque

Torque is the orthogonal component of the distance between the object and the force, multiplied by the force, resulting in a twisting moment about an axis or point. Torque has a very specific direction to it, clearly defined by popular convention (also known as the right hand rule). Torque is yet another tool use to solve these static problems, giving us another three sets of equations for equilibrium, moment (torque) about the \( x \)-, \( y \)- and \( z \)-axis.

3.1 2D Torque

In the 2-D sense, torque is the lever, multiplied by the force perpendicular to the lever-arm. As one of my teachers once described it;

"Back several decades, the people who made farm equipment would have recommended torque specifications for the various farm equipment, tractors, and stuff like that. The problem was, most of the farmers had no idea what torque was. To describe it to them, they explained it like this; if you have a 1 foot wrench, and put 1 lb on the end, you have 1 lb-ft of torque. 2lbs or a 2 foot wrench, you get 2lb-ft of torque."

-Pete Wade

3.2 Cross-Products (3d Torque)

To find a torque in three dimensions, we will use the cross product.

The Cross-Product is really just a way of finding a vector at right angles to the two other vectors.

In Linear Algebra notation, the cross product is really just a matrix made of three columns and two rows. Since the goal is to find the orthogonal component to the original two vectors, we can use cofactor expansion to show the definition of the cross product.

\[
\mathbf{A} \times \mathbf{B} = \text{Det} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}
\]

Which Expands to:

\[
= a_2b_3\hat{i} + a_3b_1\hat{j} + a_3b_2\hat{k} - a_1b_3\hat{i} - a_3b_2\hat{j} - a_2b_1\hat{k}
\]
Or, in matrix notation:

\[
\begin{pmatrix}
a_2 & a_3 \\ b_2 & b_3
\end{pmatrix}
\hat{i} - \begin{pmatrix}
a_1 & a_3 \\ b_1 & b_3
\end{pmatrix}
\hat{j} + \begin{pmatrix}
a_1 & a_2 \\ b_1 & b_2
\end{pmatrix}
\hat{k}
\]

Essentially, what we have here is two vectors being forced to show us a third vector that is orthogonal to the two we have, where \(i\) is the \(x\)-axis component, \(j\) is for the \(y\)-axis component, and \(k\) for the \(z\)-axis component. This will also define the three dimensional space we are working in, whereas the first two vectors only defined a plane. Almost exclusively, we can and will use the standard Euclidean \(x-y\) and \(z\) space. It should be said that the cross product only works in three dimensions. Otherwise, we are forced to either alter it to fit these dimensions, if it is two dimensional, or use the dot product \((a \ast b = 0)\) which holds true for any dimension.

### 3.3 Matrices

While it is possible, (and common) to solve linear equations through substitution, it is far more expedient to use linear algebra. We can compact our data into a matrix of appropriate size, and then fill it with values of the coefficients. This gives us our base, or A matrix. The answers, (the right side of our equilibrium equations), is set apart into a separate matrix, (Mx1) that we will call B. From these, we combine them, and solve. For example: we have three equations:

\[
\begin{align*}
x + 2y - 3z &= 1 \\
2x + 1y - 2z &= 3 \\
y - 5z &= 2
\end{align*}
\]

and we can place them in a matrix:

\[
\begin{pmatrix}
1 & 2 & 3 & 1 \\
2 & 1 & -2 & 3 \\
0 & 1 & -5 & 2
\end{pmatrix}
\]

### 3.4 REF and RREF

We will solve our matrices by using REF and RREF (while we can use other methods, this is more common and easier). REF, standing for Row Echelon Form, subtracts the rows to reduce matrices into a upper triangle, like this:
And from this we can use back substitution to solve for our variables. RREF, is Reduced Row Echelon Form, and it essentially does just that. It performs ref on the upper triangle, to where we have only pivots left and corresponding answers.

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

Both of these methods can break down however, if the equations are not linearly independent, if there are too few equations, or even if there are too many. Depending on which of these happens, we get a different interpretation of what is happening within the body.

## 4 Equilibrium Breakdowns

### 4.1 Series of Equations

Using the methods described earlier, we can find independent linear equations that describe mathematically what is happening in the body. When we combine these equations, we can solve them to find the individual values, and thus solve the problem. But not all bodies can be described in just six equations of forces and moments. To overcome this, we can ”cut” our body, and solve for parts of it, or if this is not possible, we can use the strength and flexibility of the materials to find more equations.

### 4.2 Statically Determinate

When our matrix of equations is not the proper size, $n$ by $n+1$, (the number of rows, one plus the number of columns) or if there are not enough linearly dependent equations, RREF will return a matrix that has one of two things:

- $0 = (a \text{ number})$ meaning: no solutions
an example matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

You can see that our matrix has the second pivot as 0 = 1. This means that our matrix has equations that are linearly dependent, linearly dependant meaning that some of our equations are just multiples of each other, and we will need more linear independent equations to solve it.

- 0=0 meaning: infinite solutions

For example, if our earlier matrix had returned something like this:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]

In our third line, we have 0 = 0, which tells us that our body is not stable, and that it will not hold up to any load without further reinforcement.
5 Example

In this example, we are trying to solve for the forces located in the beams. Since we do not have any initial conditions, we must solve for variables, which is good. With variables we can change them at will with very minimal hassal to observe the effects. We will apply the loads only on the joints. You can see that we have applied compression forces at all members, and labeled them for ease. Applying the method of joints, we get these equations, labeled 1 – 6 for the joints they are taken from:
1) \[ R_1 - \frac{1}{\sqrt{2}} A - B = F_{1x} \]
   \[ \uparrow \quad N_1 - \frac{1}{\sqrt{2}} A = F_{1y} \]

2) \[ \frac{1}{\sqrt{2}} A - B = F_{2z} \]
   \[ \uparrow \quad \frac{1}{\sqrt{2}} A + C = F_{2y} \]

3) \[ B - D = F_{3x} \]
   \[ \uparrow \quad -C = F_{3y} \]

4) \[ E - G = F_{4x} \]
   \[ \uparrow \quad F = F_{4y} \]

5) \[ D - \frac{1}{\sqrt{2}} H = F_{5x} \]
   \[ \uparrow \quad -F - \frac{1}{\sqrt{2}} H = F_{5y} \]

6) \[ N_6 + G + \frac{1}{\sqrt{2}} H = F_{6x} \]
   \[ \uparrow \quad R_6 + \frac{1}{\sqrt{2}} H = F_{6y} \]

And are entered into our matrix:
\[
\begin{bmatrix}
1 & 0 - \frac{1}{\sqrt{2}} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & - \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{\sqrt{2}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
R_1 \\
N_1 \\
A \\
B \\
C \\
D \\
E \\
F \\
G \\
H \\
N_6 \\
R_6
\end{bmatrix}
\begin{bmatrix}
F_{1x} \\
F_{1y} \\
F_{2x} \\
F_{2y} \\
F_{3x} \\
F_{3y} \\
F_{4x} \\
F_{4y} \\
F_{5x} \\
F_{5y} \\
F_{6x} \\
F_{6y}
\end{bmatrix}
\]

Now, we can solve. Using RREF we get this result:
Notice how we only have twelve variables, forming a 12 by 12 matrix. With larger and more complex structures, we have both more variables to solve for, and we have more factors to consider. In more realistic examples, such as with a building or bridge, we have to take into account each material strength, each resonance for the building, outside stresses such as wind and weather, etc. So, while it is possible to solve for each individual variable, it is much faster and easier to solve this way. In addition to that, we can assume we have less errors because we have less chances to make errors. Our chances for errors in solving this with substitution, grows increasingly likely with every substitution made.
6 Conclusion

Linear algebra is a useful tool for engineering, in how we can solve for a large number of variables in such a short time. In addition, it is interesting to note, many of the calculus theorems used in engineering classes are proved quickly and easily through linear algebra.
7 Bibliography

Sources of Information;


http://commons.bcit.ca/math/examples/civil/linear_algebra/index.html

http://en.wikipedia.org/wiki/Linear_algebra