Atmospheric Thermodynamics

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Abstract

The goal of this project is to use the fundamental laws of thermodynamics to discuss selected topics in atmospheric science; in particular atmospheric thermodynamics.
1 Introduction

Atmospheric thermodynamics serves as a foundational topic in studying meteorology. In this project, previously researched material has been taken from various texts, reorganized, and presented in an explicit way to utilize differential equations in order to describe them. There are many more topics in atmospheric thermodynamics than are introduced in this article such as the effect of water vapor in the atmosphere. However, in order to just get a basic understanding of this field it is easiest to look at a dry atmosphere. Differential equations will be used in the derivations of equations that are very useful in describing atmospheric concepts which will include

- Ideal Gas Law
- Hydrostatic Equation
- Barometric Formula
- Geopotential
- The First Law of Thermodynamics
- Specific Heat
- Enthalpy
- Adiabatic Process
- Air Parcels
- Dry Adiabatic Lapse Rate
- Static Stability
2 Basic Equations

In this section it will be shown how the ideal gas law can be the foundation for showing how pressure, volume, and temperature are related. Next will be shown how the hydrostatic equation can explain the balancing forces acting on a thin slab of air in the atmosphere. Following, an expression will be derived for barometric pressure as a function of height and then solved.

2.1 Ideal Gas Law

The first assumption to be made is that atmospheric gases, whether considered individually or as a mixture, obey the ideal gas law exactly,

\[ PV = nR^*T \]  

(1)

where \( P \) is pressure with units Pascals, Pa, \( V \) is the volume with units \( m^3 \), \( T \) is the temperature with units Kelvin, K, and \( n \) is the number of moles of a gas; where a mole is a unit of measurement for the amount of substance. The weight of atoms and molecules are given in grams per mole. The number of molecules or particles in a mole was experimentally verified by Avagadro to be \( 6.2 \times 10^{23} \). \( R^* \) is the universal gas constant and has the value of \( 8.314 \text{J mol}^{-1} \text{K}^{-1} \). There are multiple forms of this law that are useful in meteorology; however, due to the extent of the topics discussed in this paper, only one other form is used. Examining the relationship of these physical properties in dry air (excluding \( \text{H}_2\text{O} \)) one has,

\[ P_d\alpha_d = R_dT \]  

(2)

Where \( P_d \) is the pressure of dry air, \( \alpha_d \) is the specific volume of dry air or the volume occupied by 1 kg of dry gas. \( R_d \) is the constant for dry air and is defined as \( 1000R^*/M_d \), where \( M_d \) is 28.97 g mol\(^{-1}\), so \( R_d = 287.0 \text{J mol}^{-1} \text{kg}^{-1} \).

2.2 Hydrostatic Equation

The hydrostatic equation shows how forces acting on a thin slab of air in the atmosphere essentially balance so long as there is no accelerating forces acting on the slab other than gravity. This is due to the fact that pressure decreases with height in the atmosphere, so the air below with greater density, exerts forces between the molecules in the slab and the underlying air creating an upward force. This is actually attributed to Archimedes’ principle
for fluid dynamics, which states that a body immersed in a fluid experiences an upward force equal to the weight of the displaced fluid. This is counterbalanced by a downward force equal to the acceleration due to gravity times the density of the slab of air pushing down on it. Mathematically this can be expressed by using Newton’s second law of motion which states

\[ \Sigma F(x, y) = ma \]

In this case there are no horizontal forces and no acceleration. Plugging in the forces, the hydrostatic equation can be derived and solved as follows,

\[ \Sigma F(y) = 0 \]

\[ -\Delta P - g\rho \Delta z = 0 \]

The negative in front of \( \Delta P \) is there because pressure decreases with height which makes this a positive force.

\[ \Delta P = g\rho \Delta z \]

Letting \( \Delta z \to 0 \) gives

\[ -dP = g\rho dz \quad (3) \]

\[ -\int_{P(z)}^{P(\infty)} dP = \int_{z}^{\infty} g\rho dz \]

Where \( P(\infty) = 0 \), making the final expression,

\[ P(z) = \int_{z}^{\infty} g\rho dz \]

This says that at height \( z \) in the atmosphere, the pressure acting on the slab is the weight of the vertical column of air above it. This equation allows one to show that the standard pressure at sea level \( z = 0 \) is \( 1.013 \times 10^5 \) Pa.

### 2.3 Barometric Formula

The goal here is to derive an ordinary differential equation which when solved, will show a closed form solution to how pressure changes in the atmosphere as a function of height. Some assumptions will be made in order to simplify the model: (i) the layer in the atmosphere is only composed of dry air,
the atmosphere is in hydrostatic balance, (iii) without having to say anything about temperature lapse rates which will be covered later, one will assume that the temperature throughout the layer is constant, or isothermal.

Beginning with the ideal gas law (1), with \( n = \frac{m_d}{M_d} \), recall that \( m_d \) is the total mass of the dry air and \( M_d \) is the molar mass of dry air (0.02896 kg/mol). Also \( \rho = \frac{m_d}{V} \), so making these substitutions puts the ideal gas law in the form,

\[
P = \frac{\rho R^* T}{M_d}
\]

Again, assuming the atmosphere is in hydrostatic balance, dividing (3) by (1),

\[
\int \frac{dP}{P} = -\int \frac{gM_d}{R^* T} dz
\]

This is an autonomous, first order differential equation that says that pressure decreases (negative sign) as a function of height in the atmosphere. Since the rest of the terms are constants, one can simply integrate both sides to get,

\[
\ln(P) = -\frac{gM_d}{R^* T} z + C
\]

where \( C \) is the constant of integration. Solving for pressure and simplifying,

\[
P(z) = Ae^{-\frac{gM_d}{R^* T} z}
\]

where \( A = e^C \). Normally, when doing this algebraic maneuver, one would put absolute values around the \( P(z) \) since there are two possible results (positive and negative); however, since the initial condition has \( P(0) > 0 \) one can ignore this; that is \( P(z) \) will always be positive. In order to solve for \( A \), an initial value problem or an initial condition must be established. In this case one can define pressure at altitude zero to be \( P_0 \). That is \( P(0) = P_0 \). Solving for \( A \) by setting \( z = 0 \) yields,

\[
P = A = P_0
\]

So, the final expression called the barometric formula can be written as follows,

\[
P(z) = P_0 e^{-\frac{gM_d}{R^* T} z}
\]

The figure shown below was made on Matlab using dfield8 to show a number of solutions to this differential equation given different initial conditions or
initial value problems. The red line is the solution to the barometric formula with an initial condition set at standard temperature and pressure (STP).

![Figure 1: Solutions to the barometric formula](image)

2.4 Geopotential

Begin with the notion that work is required to raise a mass of 1 kg from sea level to height $z$ in the atmosphere. This work is called geopotential $\Phi$ with units of $J \text{ kg}^{-1}$ or $m^2/s^2$. The force acting on this mass is approximately $g$, and the work required to lift the mass from $z$ to $z + dz$ is $gdz$,

$$d\Phi = gdz = -\alpha dP = -\frac{1}{\rho}dz$$

(4)

$$\int_0^\Phi d\Phi = \int_0^z gdz$$
This is important, because the geopotential only depends on height $z$ in the atmosphere and is independent of the path the mass took to get to that level.

3 The First Law of Thermodynamics, Adiabatic Processes, Air Parcels, Dry Adiabatic Lapse Rate

3.1 First law of Thermodynamics

The first law of thermodynamics is a statement about the energy of a system. First, there is internal energy due to the microscopic interaction of molecules in the system in the form of kinetic and potential energy. The increase of internal kinetic energy also causes the temperature of the system to rise, while the potential energy is affected by the relative positions of the molecules with each other and the forces between them. Suppose a closed system takes in heat $q$ via conduction or radiation and consequently does external work $w$ on the environment. The excess in energy of the system, i.e. the energy not used to do external work, increases by the difference $q - w$. Due to the conservation of energy, the change in the internal energy from an initial state to a final state is path independent just like geopotential and can be written as follows,

$$u_2 - u_1 = q - w$$

In differential form,

$$du = dq - dw$$  \hspace{1cm} (5)$$

Work is defined as force times displacement and has the sign convention as follows. If the final volume of the system is less then the initial volume then the environment has done work on the system and work is said to be negative,

$$V_2 < V_1 \rightarrow w < 0$$

Adversely, if the final volume is more than the initial volume then the system has done work on the environment and work is said to be positive,

$$V_1 < V_2 \rightarrow w > 0$$
Therefore, one can write an expression for work as follows,

\[ dw = Fdx = PAdx = PdV \]  \hfill (6)

\[ w = \int_{V_1}^{V_2} PdV \]

The first law can be rewritten by replacing \( PdV \) with \( Pd\alpha \) as

\[ du = dq - Pd\alpha \]  \hfill (7)

ONLY IF one is dealing with a unit mass. This makes it possible to use (2) in further derivations.

3.1.1 Specific Heat

Specific heat is a value used to show how a material changes temperature when heat is added to the material (system). The temperature change is \( T + dT \) and the specific heat at constant pressure or volume is,

\[ c_P = \frac{dq}{dT} \]  \hfill (8)

and,

\[ c_V = \frac{dq}{dT} = \frac{du}{dT} \]  \hfill (9)

3.1.2 Enthalpy

Enthalpy is simply a preferred expression of a systems energy, because it simplifies certain descriptions of energy transfer. This is because a change in enthalpy takes into account the energy transferred to the environment through the expansion of the system. This will be very useful when discussing adiabatic lapse rates. Enthalpy can be derived as follows,

\[ q = u - w \]

\[ \Delta q = (u_2 - u_1) + P(\alpha_2 - \alpha_1) \]

\[ \Delta q = (u_2 + P\alpha_2) - (u_1 + P\alpha_1) \]

\[ \Delta q = h_2 - h_1 \]
Where $h \equiv u + P\alpha$ or in differential form,

$$dh = du + d(P\alpha)$$

If volume is constant,

$$dq = dh - \alpha dP$$

Comparing (8) for constant pressure with (10),

$$dh = c_p dT$$

Integrating both sides,

$$\int_0^h h = \int_0^T c_p dT$$

$$h = c_p T$$

Using the concept of enthalpy, one can look at a layer of air in the atmosphere assuming it is in hydrostatic balance, and can say something about the enthalpy of the layer. As the layer receives heat in the form of solar radiation at constant pressure the enthalpy of the layer increases. This increase is commonly referred to in meteorology as *sensible heat*. Of the energy from solar radiation, part is used by heating up the layer and the rest is used to expand the layer at constant pressure, so

$$Pd\alpha = RdT.$$  \hspace{1cm} (12)

This is a statement that says the system (*i.e.* the layer of air) does work on the overlying air.

### 3.2 Adiabatic Process, Air Parcels

#### 3.2.1 Adiabatic Process

An adiabatic process is one in which the net heat energy of a system does not change (*i.e.* $dq = 0$). In an adiabatic process, pressure, volume, and temperature are free to change, so long as the $dq = 0$. Adiabatic compression is that in which the environment does work on the system. Therefore, the decrease in volume and the increase in pressure results in the heating of the system. Alternatively, adiabatic expansion is that in which the system does work on the environment. Therefore the increase in volume and the decrease in pressure results in the cooling of the system. The area under an adiabatic curve on a pressure-volume (P-V) diagram is defined as work and is shown below.
3.2.2 Air Parcels

When thinking about a thermodynamic process in the atmosphere, it is useful to think of the thermodynamic system as an air parcel, or a dimensional piece of air that can range in size horizontally from millimeters, to the scale of the earth. For the purpose of this project, again it is only necessary to consider an air parcel with a unit mass of dry ideal gas, which has a volume of approximately $0.7734 \text{ m}^3$ at STP. Some assumptions should be made however to simplify the model. In reality, one or more of these assumptions can’t be applied, but for the sake of this model one can say: (i) This parcel is insulated from its environment so its temperature changes adiabatically with vertical displacement, (ii) the parcel is always adjusting its pressure to match that of the surrounding environmental air at the same level, (iii) the atmosphere is assumed to be in hydrostatic balance and finally, (iv) the parcel is moving slowly enough that its macroscopic kinetic energy can be neglected. Utilizing (4), (10), and, (11), one can put together a more general expression for a parcel as it rises or sinks in the atmosphere,

$$dq = dh + d\Phi = d(c_pT + \Phi)$$

3.3 Dry Adiabatic Lapse Rate, DALR

Recall (2) is the ideal gas law when dealing with 1 kg of dry air. Differentiating this requires the chain rule remembering that $R_d$ is a constant,

$$dP_d\alpha_d + P_d d\alpha_d = R_d dT \quad (13)$$
Utilizing (5) and knowing that in an adiabatic process $dq = 0$, one has $du = -dw = -c_vdT$. From (12),

$$-dw = -c_vdT = P_d d\alpha_d$$  \hspace{1cm} (14)

So combining (14) with (13),

$$-c_vdT + dP_d \alpha_d = R_d dT$$

Rearranging the terms to get $dT$ on the right hand side and doing some algebra

$$dP_d \alpha_d = dT(R_d + c_v)$$

Where $R_d = c_p - c_v$ so,

$$dP_d \alpha_d = c_p dT$$

$$\frac{dT}{dP} = \frac{\alpha_d}{c_p}$$

Using the hydrostatic equation that says $dP/\alpha d = -g \rho$ where $\rho = m_d/V$, $\frac{dP}{\alpha d} = -g \frac{m_d}{\alpha d}$, therefore,

$$\frac{dT}{d\alpha} = \frac{dP_d}{\alpha d} \frac{dT}{dP_d} = -g \frac{m_d}{\alpha_d} \frac{\alpha_d}{c_p} = -g \frac{m_d}{c_p}$$

Now we can define the dry adiabatic lapse rate, DALR, $\Gamma_d$ as,

$$\Gamma_d \equiv -\frac{dT}{d\alpha} = g \frac{m_d}{c_p}$$

For a unit mass,

$$\Gamma_d = \frac{g}{c_p}$$

The figure shown below was generated on Matlab using Dfield8 and the red highlighted line is the solution to the DALR with an initial condition set at STP.
Figure 3: Solutions to the DALR

4 Static Stability

In this section, first an expression will be derived that shows how a dry air parcel accelerates upward in the atmosphere when the lapse rate $\Gamma_d$ is greater than the lapse rate of the surrounding (environmental) air $\Gamma$. This is due to the fact that the environmental lapse rate is for a stationary atmosphere, whereas the DALR is for an adiabatic process. Following the expression for upward acceleration will be the derivation of the subsequent vertical displacement that results by a restoring force acting on the parcel. This latter expression can be solved using Euler’s formula which is explained throughout.
4.1 Upward acceleration

First, consider an unsaturated parcel with density $\rho'$ and temperature $T'$ where the density and temperature of the ambient (surrounding air) is $\rho$ and $T$ respectively. From the free body diagram, one can see there are two forces acting on the parcel. The force in the negative direction is the density of the dry air times the acceleration due to gravity. The force in the positive direction is the density of the underlying ambient air times the acceleration due to gravity; again, this is from Archimedes’ principle. From Newton’s second law, one can sum the forces acting on the parcel like so,

$$\sum F = ma$$

$$\rho g - \rho' g = ma$$

Doing some algebra and making the substitutions $m = \rho'$ and $a = \frac{d^2z'}{dt^2}$,

$$\frac{g(\rho - \rho')}{\rho'} = \frac{d^2z'}{dt^2}$$

(15)

It is more efficient to have this expression in terms of $T$ and $T'$ since rawindsonde (weather balloon) data provides temperature data. From the ideal gas
law in the form \( p = \rho RT \), it is apparent that \( \rho \) is inversely proportional to temperature, so making the substitutions \( \rho = 1/T \) and \( \rho' = 1/T' \) into (15) and doing some algebra,

\[
\frac{d^2 z'}{dt^2} = g \left( \frac{T' - T}{T} \right)
\]  

(16)

### 4.2 Subsequent Vertical Displacement

Now, since the parcel displaced is more dense than the surrounding air because \( \Gamma_d > \Gamma \) as explained earlier, a restoring force tries to return the parcel to its original equilibrium position. The greater the difference between \( \Gamma_d \) and \( \Gamma \), the greater the restoring force. Begin by calling the equilibrium position \( z = z_0 \) and the distance displaced is \( z' \). Therefore \( z' = z - z_0 \). The surrounding air temperature will be called \( T_0 \) and the temperature of the parcel \( T' \) after being lifted distance \( z' \) is \( T' = T_0 - \Gamma_d z' \). Since \( T_0 \) was the temperature of the surrounding air at \( z_0 \), one can say the the difference from \( T_0 \) to \( T \) (the new surrounding temperature) is proportional to the difference between \( \Gamma_d \) and \( \Gamma \). This can be written mathematically as,

\[
T' - T = -(\Gamma_d - \Gamma)z'
\]

Substituting this into (16),

\[
\frac{d^2 z'}{dt^2} = -g (\Gamma_d - \Gamma)z'
\]  

(17)

It is useful to make a substitution \( N = \left[ \frac{g}{T_0} (\Gamma_d - \Gamma) \right]^{1/2} \) \( N \) is called the Brunt-Vaisala frequency and if \( \Gamma_d > \Gamma \) then \( N \) must be a real number, and \( N^2 \) must be greater than zero, hence the one half power in the expression for \( N \). Therefore, substituting in \( N \) into (17), one has a homogeneous, autonomous, second order differential equation,

\[
\frac{d^2 z'}{dt^2} + N^2 z' = 0
\]  

(18)

In order to solve (18) for \( z' \) one must explore Euler’s formula for solving second order differential equations. He begins with a very good guess for, in this case \( z' \). The guess is that \( z' = e^{\lambda t} \). Taking first and second derivatives,

\[
\frac{dz'}{dt} = \lambda e^{\lambda t}
\]
\[ \frac{d^2 z'}{dt^2} = \lambda^2 e^{\lambda t} \]

Substituting these values into the original equation, ignoring the first derivative (since there is no first derivative in the original ODE),

\[ e^{\lambda t}(\lambda^2 + N^2) = 0 \]

Since \( e^{\lambda t} \) can’t ever equal zero, one has \( (\lambda^2 + N^2) = 0 \)

This is a second degree polynomial and can be solved easily using the quadratic formula

\[ \lambda = -b \pm \sqrt{b^2 - 4ac} \]

Where \( a, b, c \) are the coefficients in the polynomial written in the form \( \lambda(t) = a\lambda^2 + b\lambda + c \). Utilizing the quadratic formula and solving it,

\[ \lambda = \pm N i \]

This gives two roots, \( \lambda_1 = Ni \), \( \lambda_2 = -Ni \). From Euler’s formula, \( e^{i\theta} = \cos \theta + i \sin \theta \), one knows that in the case of having two complex roots, taking the real and imaginary parts as solutions to (18), one has

\[ z' = c_1 \cos(\lambda_1 t) + c_2 \sin(\lambda_2 t) \]

There are two initial conditions to solve for the constants \( c_1 \) and \( c_2 \). The first is that at time \( t = 0 \) \( z' = z'(0) \). Plugging in zero for \( t \) into the general solution, one gets

\[ z'(0) = c_1 = z' \]

The second initial condition is for velocity, saying that \( \frac{dz'}{dt} = 0 \). Taking the derivative of the general solution yields,

\[ \frac{dz'}{dt} = -c_1 N \sin(NT) + c_2 N \cos(NT) \]

Plugging in the initial condition and solving for \( c_2 \),

\[ \frac{dz'}{dt}(0) = c_2 N = 0 \]
So $c_2 = 0$. This makes the final closed form solution to the second order ODE,

$$z'(t) = z'(0) \cos(Nt)$$

This says that when the parcel is displaced vertically, it’s subsequent vertical displacement is an oscillatory motion called *buoyancy oscillation* about its equilibrium level $z$ with an amplitude equal to its initial displacement $z'(0)$ with a frequency $N$. Again $N$ is essentially looking at the difference between $\Gamma_d$ and $\Gamma$. The greater the restoring force which is proportional to this difference, the greater the frequency, and the greater the static stability.

![Graph showing position and velocity vs time](image)

*Figure 5*
5 Conclusion

To summarize, it has been shown how the knowledge of differential equations is essential to understanding even the most basic concepts in atmospheric thermodynamics. The processes introduced in this project had clear and consistent assumptions that helped simplify the models derived above. Although some of the assumptions cannot be applied to the actual atmosphere to make more accurate models, they are assumptions that enable one to look at these processes with knowledge limited to differential equations, basic chemistry, and Newtonian physics. Since all of the models used assumed that the atmosphere was only consisting of dry air (excluding H₂O), they only describe a limited number of phenomena; however, the assumption serves as a good way of simplifying the models. Nonetheless, this project showed that the above stated knowledge makes it possible to study such a field with little instruction besides reference texts. The next step would be introducing other topics in atmospheric thermodynamics; most of which in-
clude an atmosphere with water vapor as well as saturated air. With more time, these further topics could be looked at using the same fundamental knowledge of differential equations.

6 Appendix

6.1 Matlab m-files

%Defining parameters
function y=z(t)
global N
global g
global T
global r_d
global r
global y_0

N=(((g/T)*(r_d-r))^(1/2));
g=9.8;
T=273;
r_d=9.8;
r=6.49;
y_0=2;
y=y_0*cos(N*t);

%System of equations
function xprime=z
xprime=zeros(2,1);
xprime(1)=x(2);
xprime(2)=-N^2*x(1);

%Executing plots
function ArlanODEsysproject
[t,x]=ode45(@z,[0,100],[2,0]);
plot(t,x(:,1),’r’,t,x(:,2),’k’);
legend(’z’,’dz/dt’)
```
title('Position and Velocity vs. Time')
xlabel('t')
ylabel('z,dz/dt')
zlabel('t')
figure
plot(x(:,1),x(:,2))
title('Velocity vs. Position')
xlabel('z')
ylabel('dz/dt')
figure
plot3(t,x(:,2),x(:,1))
title('Time vs. Velocity vs. Position')
xlabel('t')
ylabel('dz/dt')
zlabel('z')
```
References


