Differential Equations of Glacial Proportions

Katie Davis, Sally Kintner

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Abstract

Though glacier dynamics are quite complicated and dependent on many parameters, the relationship between a glacier's area and volume can be studied using a simplified model. Using data from 1970-1979 for the South Cascade Glacier in Washington State, this relationship is analyzed with both a first order and a second order system of ordinary differential equations.
1 Introduction

Glaciers are amazingly dynamic systems. Their very definition, as perennial masses of ice that flow, encompasses movement. Glaciers are comparable in many ways to rivers of ice, and they have often been modeled using derivations of the standard equations for fluid motion. But with their slow flow and the viscoelastic properties of ice under stress, glaciers are unique, and the best way to model them is still under investigation [3, 6].

With the increasing attention to global climate change, glaciology has become ever more important. Predictions for the melting of glaciers and ice sheets has vast implications for sea level rise, local climate effects, and the clean water supply of billions of people worldwide[1, 2, 7].

Many scientists have turned to glaciers not only as worthy of investigation in their own right, but particularly as barometers of the current climate change trajectory of the world[2]. The slow but steady response of a glacier to changes in climate is a better indicator of trends than is, say, vegetation, which reacts more immediately to inputs of precipitation or sunlight intensity in a given season. As Earth’s climate warms, the need for such reliable barometers as ice sheets and glaciers becomes more and more clear. The ability to characterize a glacier’s response to climate would represent a sort of calibration of this barometer. Information available through a glacier could be critically relevant both for the sake of predicting possible futures as well as understanding changes that have already begun to occur.

The search for a precise model, then, is a serious one. In recent decades, several models have taken shape, including the widely used Glen Flow Law and the Shallow Ice Approximation[3]. However, as Ralf Greve and Heinz Blatter, the authors of a definitive text on ice dynamics, note, this field of study is only beginning to gain attention and weight. The laws governing flow and mass balance can only benefit from the testing of various types of models.

This paper, which investigates an approach put forward by Harrison et al.[4], attempts to cast light on what can be learned from an approach to glacier dynamics based on ordinary differential equations. We follow their method of setting up a system of first-order equations relating volume and area, then deriving a set of second-order equations to further analyze their characteristics. Much can be learned from this approach, which is vastly simpler than some of the “traditional” models[1].
Figure 1: South Cascade Glacier, Washington, 1970
2 Development of Equations

2.1 Parameters

A great many parameters go into the dynamics of a glacier. Both interior and exterior forces and inputs go into the net changes a glacier may undergo. Flow rate, deformation, water output, area and volume are all functions of a long list of variables including (but not limited to) precipitation, bedrock surface, sun exposure, etc.

Harrison et al. simplify their model by quantifying many of these inputs in constant rates. The derivation and usefulness of these simplifications is detailed in papers by Elsberg et al. (the team of authors is nearly the same as that of the Harrison et al. study)[1] and Johannesson et al.[5]. The three primary parameters we considered are: the area time-scale $\tau_A$, a thickness scale $H$, and an initial reference condition $A_0$ which quantifies the amount by which area is out of balance with the initial volume. Listed below, in Table 1, are our parameters and the values we used in our analysis. The numerical values used, also derived by Harrison et. al [4], come from a data set on South Cascade Glacier in Washington state from 1970-1997. Use of these approximations both enables the ordinary differential equations approach and, as Elsberg et al. argue, improves upon traditional methods [1].

Table 1.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_A$</td>
<td>area time-scale</td>
<td>8.0 years</td>
</tr>
<tr>
<td>$H$</td>
<td>thickness scale</td>
<td>123 m</td>
</tr>
<tr>
<td>$A_0$</td>
<td>initial reference area</td>
<td>$0.094 \times 10^6$ m$^2$</td>
</tr>
<tr>
<td>$b$</td>
<td>effective balance rate at terminus</td>
<td>5.5 m/yr</td>
</tr>
<tr>
<td>$g$</td>
<td>effective of balance rate gradient with elevation</td>
<td>0.024/yr</td>
</tr>
<tr>
<td>$B$</td>
<td>a balance rate dependent on climate</td>
<td>0</td>
</tr>
<tr>
<td>$\tau_V$</td>
<td>volume time-scale defined by $\frac{1}{(b/H)-g}$</td>
<td>48 years</td>
</tr>
</tbody>
</table>

In our equations and analysis, we use $A$ and $V$ as our area and volume variables. It is important to the analysis to understand that these are not quantifying actual glacier surface area and volume. They are representations of the change of area and volume from an initial state at time $t = 0$ to be $A = 0$ and $V = 0$.

2.2 First Order Equations

Let us first explore the relationship of area to volume. Both are functions of time; however, area adjustment can also be taken as a dependent function of volume. Our first equation proceeds from taking the rate of adjustment of area to be proportional to the amount that it is out of adjustment with volume.
This idea is made more clear by visualizing a glacier which loses a significant chunk of its volume to calving (a sudden brittle separation of a piece of ice at a glacier’s terminus). The ice, which acts visco-plastically in a slow-flowing glacier, “wants” to have an even gradient with the thinnest ice near the terminus. It will begin to deform towards adjusting for the lost section, and area will return to a “balance” with volume. Importantly, the rate at which this deformation occurs will be in proportion to the amount of difference of this area to the “proper” area. In mathematical terms,

\[ A' = \frac{-1}{\tau_A} (A - A_a(V)), \]

where \( A \) represents an instantaneous area and \( \tau_A \) is a factor of proportionality. The term \( A_a(V) \) is the area’s difference from an adjusted reference state.

The area-adjusted state is now approximated by a thickness scale \( H \), where \( 1/H \), approximated using an infinite series [4], is roughly equal to the derivative of this area-adjustment with respect to volume. Adding an initial condition for the area, \( A_0 \), we arrive at our first equation[4]:

\[ A' = \frac{1}{\tau_A H} V - \frac{1}{\tau_A} A - \frac{A_0}{\tau_A} \]  

To derive a second equation about the rate of change of volume, Harrison et al. make a simple statement of mass continuity,

\[ V' = gV - bA + B \]  

where \( b \) is the balance rate at the terminus, \( g \) is how this rate is distributed over the entire glacial surface, and \( B \) is an external balance rate that can be generalized as a climatic forcing term.

### 2.3 Second Order Equations

From the two-first order ODEs previously described, it is possible, for analytical reasons, to separate \( A \) and \( V \) into two second-order ODEs. By solving Equation (1) for \( V \) and substituting into Equation (2), we get the second-order equation for area:

\[ \tau_A A'' + (1 - g\tau_A)A' + \frac{1}{\tau_V} A = \frac{B}{H} + gA_0 \]

Solving this for \( A'' \), we get a linear second-order equation only in terms of \( A \):

\[ A'' = (g - \frac{1}{\tau_A})A' - (\frac{1}{\tau_V\tau_A})A + \frac{B}{H\tau_A} + g\frac{A_0}{\tau_A} \]  

[4]
By solving Equation (2) for $A$ and substituting into Equation (1), we get the second-order equation for volume:

$$\tau_A V'' + (1 - g\tau_A) V' + \frac{1}{\tau_V} V = B + \tau_A B' + bA_0$$

Solving this for $V''$, we get a linear second-order equation only in terms of $V$:

$$V'' = (g - \frac{1}{\tau_A})V' - \left(\frac{1}{\tau_V\tau_A}\right)V + \frac{B}{\tau_A} + B' - \frac{b}{\tau_A}A_0$$

(4)

To simplify these equations, Harrison et al have introduced a volume time scale parameter, $\tau_V$, such that:

$$\tau_V = \frac{1}{bH - g}$$

3 Analysis

We have seven parameters (see Table 1) to consider in our analysis. All of these terms, with the exception of $B$, were experimentally determined from observations of the South Cascade Glacier in Washington state from 1970-1997 and calculated through methods laid out by Elsberg et al[1]. Harrison et al used data to make a comparison with more traditional models. For our initial comparison analysis of the first- and second-order systems, we assume climate effects to be minimal and constant, making $B = 0$.

3.1 First Order System Analysis

Analysis of the first-order system can be done using matrix math. The system in matrix form is as follows.

$$\begin{bmatrix} A \\ V \end{bmatrix}' = \begin{bmatrix} -\frac{1}{\tau_A} & \frac{1}{\tau_V} \\ -b & 0 \end{bmatrix} \begin{bmatrix} A \\ V \end{bmatrix} + \begin{bmatrix} -A_0 \\ \tau_A B \end{bmatrix}$$

(5)

We take the homogeneous case ($-A_0/\tau_A = 0, B = 0$) and enter known values to yield

$$\begin{bmatrix} A \\ V \end{bmatrix}' = \begin{bmatrix} -0.125 & 0.001 \\ -5.5 & 0.024 \end{bmatrix} \begin{bmatrix} A \\ V \end{bmatrix}$$

(6)

This system has complex eigenvalues $\lambda = -0.505 \pm 0.0073i$. 
Linearization of the system can also be undertaken. Computation of the Jacobian for the inhomogeneous case leads to a matrix identical to that in Equation (5), with constant terms. A single $V$ nullcline $A = (gV - B)/b$ and an $A$ nullcline $V = H(A_0 + A)$ intersect once, at equilibrium point $(2.496 \times 10^7, 1.089 \times 10^5)$ to be investigated. Trace-determinant analysis of this matrix (whose trace $= g - 1/\tau_A$ and determinant $= -g/\tau_A + b/\tau_A H$ shows $T^2 - 4D = -1.56 \times 10^{-4} < 0$, with $T < 0$; this indicates a spiral sink near this equilibrium point\[8].

Use of Matlab’s numerical solver pplane8 clearly bears out this finding. All solutions plotted in Figure 1 dive eventually to the equilibrium point as expected.

Figure 2: PPlane8 showing equilibrium point at $(2.50 \times 10^6, 1.089 \times 10^5)$
Figure 2 shows a single solution curve forward in time from initial conditions \( A = 0, V = 0 \). This curve will be useful for comparison to the second-order approach described later in the paper.

\[
\begin{align*}
V' &= -bA + gV \\
A' &= -\frac{A}{T} + \frac{V}{TH} - \frac{I}{T}
\end{align*}
\]

\( T = 8 \)
\( I = 0.094 \times 10^6 \)
\( H = 123 \)
\( g = 0.024 \)
\( b = 5.5 \)

Figure 3: Single solution curve forward in time from initial conditions \((0,0)\)

Finally, it is only human to want to see the individual area and volume curves plotted with respect to time. In Figures 3 and 4 we can see solution curves representing the adjustment of each parameter. These help to understand the rapidity of adjustment as well as fluctuations that occur before the system reaches equilibrium.
3.2 Second Order System Analysis

3.2.1 Analysis of the Homogeneous Case

Because our second order equations are linear, it can be useful to analyze the homogenous case:

$$\tau_A A'' + (1 - g\tau_A) A' + \frac{1}{\tau_V} A = 0$$
and
\[ \tau_A V'' + (1 - g\tau_A) V' + \frac{1}{\tau_V} V = 0 \]

Since the coefficients of both the area and the volume homogenous cases are the same, we will use the first for area for our analysis.

To solve for roots, we first set up the characteristic polynomial for the equation:
\[ \tau_A \lambda^2 + (1 - g\tau_A) \lambda + \frac{1}{\tau_V} = 0 \]

To solve for the characteristic roots of our homogenous case, we use the quadratic equation:
\[ \lambda = \frac{- (1 - g\tau_A) \pm \sqrt{(1 - g\tau_A)^2 - 4(\tau_A)(\frac{1}{\tau_V})}}{2\tau_A} \]

Substituting the given values for our parameters yield roots of:
\[ \lambda = -0.0505 \pm 0.0073i \]

which will be recognized as corresponding to the eigenvalues found for the first order system when analyzing its homogenous case. These values suggest that the solution to our system of second-order equations will give us a spiral sink.

### 3.2.2 Comparison to Classic Mass Spring Model

One might wonder why it might be important to separate the two first-order equations into two second-order equations. One reason would be to make a comparison to the classic mass-spring model, one with which any physics student would be familiar, the general equation of which is given by:

\[ my'' + \mu y' + ky = F(t) \]

where
\begin{align*}
    m &= \text{mass} \\
    \mu &= \text{dampening constant} \\
    k &= \text{spring constant}
\end{align*}

and
\[ F(t) = \text{driving forces} \]

Our second-order area and volume equations,
\[ \tau_A A'' + (1 - g\tau_A) A' + \frac{1}{\tau_V} A = \frac{B}{H} + gA_0 \]
enable us to compare $\tau_A$ to $m$, $(1 - g\tau_A)$ to $\mu$, and $1/\tau_V$ to $k$. The driving forces for area are $B/H + gA_0$. In our analysis, since we consider $B$ to be 0, the driving forces for $A$ are only dependent on $g$ and $A_0$. Thus, changes in area are driven by the initial imbalance of area to volume and the gradient of balance rate with respect to elevation. The driving forces for volume are $B + \tau_AB' + bA_0$. In our analysis, we consider both $B$ and $B'$ to be constant and 0 so the driving forces of $V$ are dependent on $b$ and $A_0$ only. Thus, changes in area are also driven by the initial imbalance of area to volume as well as the balance rate at the terminus.

From these comparisons, we see that $A_0$ is the common driving force for both area and volume change. This makes a lot of sense as our equations model the changes in area and volume in order to achieve an area/volume balance. $A_0$ is the amount by which area is initially “out of balance” with volume.

### 3.2.3 Graphing the Second Order System of Equations

To analyze the system of second order equations from equations (3) and (4), we use Matlab’s ode45 solver to obtain numerical approximations for these equations. The initial conditions we used are:

\[
A(0) = 0 \\
V(0) = 0 \\
A'(0) = -\frac{A_0}{\tau_A} \\
V'(0) = B
\]

From these numerical solutions, we can look at the area and volume time curves.
From Figures 5 and 6, we can see both area and volume approaching their equilibrium points: $1.09 \times 10^5 \text{m}^2$ and $2.5 \times 10^6 \text{m}^3$ respectively.

It can also be helpful to look at these graphs side by side with the $A'$ vs time and $V'$ vs time, as in Figure 7.
Figure 8: Comparison of $A$, $V$, $A'$, and $V'$ vs time curves

We can see that the area initially decreases and later increases to the equilibrium point. For this model, area is dependent on volume. Therefore, area adjusts to balance with volume. $A_0$, which measures the amount by which area is out of adjustment with volume, begins at $0.094 \times 10^6 \text{m}^2$, a large positive number. The first relatively rapid drop on the area curve is a response to the wide difference between this initial state and that of volume, which we have taken at 0. Only after this initial adjustment does area slow its change and increase toward equilibrium.
Perhaps the most interesting graph of our solutions of the second order system is the plot of volume change vs area change.

![Graph of Area Change vs Volume Change](image)

Figure 9: Volume Change vs Area Change

This graph clearly shows the expected spiral sink reaching equilibrium at $(2.5 \times 10^7, 1.09 \times 10^5)$.

4 Conclusion

The value of a mathematical model is in its balance of simplicity and complexity. Models must use enough approximations to be approachable, yet retain enough detail to be realistic. Analysis of a glacier’s mass balance with an ordinary differential equations approach necessitates much approximation, but it can be very useful for understanding real-world applications at a basic level. This simplified model was able to show mass balance dynamics clearly: the tendency of area and volume to an equilibrium point, an idea of the time-scale over which this point might be reached, and the interaction of the variables as they moved toward this state.

Unfortunately, the scope of this study did not include comparison of data from a different glacier, in order to comment on the model’s utility beyond South Cascade Glacier. Elsberg et al. [1] did perform a comparison of this model with
the "traditional" approach; however, their comparison used only South Cascade Glacier as its basis. An important next step in appreciating the model’s validity would be to calculate parameters and run a model for another ice mass.

Another obvious course of continued study would be the effects of the climate term, $B$. This project was only able to hold $B$ as a zero term and investigate the glacier’s internal balance in the absence of climate input. Mathematically, this is an important foundation to further study; in real world terms, however, it has little meaning. Climate changes with time, whether over the course of single seasons or many millenia. These shifts are related to temperature, precipitation, and light intensity. It would be another mathematical puzzle to attempt to quantify such changes. As it stands, however, we have been able to show many important relationships in glacier dynamics.

Studies are ongoing to determine the accuracy of the many models so far put forward; accurate models are becoming ever more important as the world continues to investigate the pace and effects of global climate change. It remains to be seen whether this model, or the more complicated approaches that are currently more widely used, may be favored. What is certain is that such study will continue to be increasingly relevant.

5 Appendix

Matlab M-file for Second Order Analysis:

```matlab
function Glacier_comb2
[t,x]=ode45(@glacier12,[0,100],[0;(-0.094*10^6)/8;0;0]);

%Plot A vs t
plot(t,x(:,1),'r')
title('Area Change vs Time')
xlabel('Time')
ylabel('Area Change')
figure

%Plot V vs t
plot(t,x(:,3))
title('Volume Change vs Time')
xlabel('Time')
ylabel('Volume Change')
figure

%Plot A' vs t
plot(t,x(:,2))
title('Rate of Area Change vs Time')
xlabel('Time')
```

14
ylabel('Rate of Area Change')
figure

% Plot V' vs t
plot(t,x(:,4))
title('Rate of Volume Change vs Time')
xlabel('Time')
ylabel('Rate of Volume Change')
figure

% Plot A vs V
plot(x(:,3),x(:,1),'b')
title('Area Change vs Volume Change')
xlabel('Volume Change')
ylabel('Area Change')
figure

% Plot A vs t, V vs t, A' vs t, V' vs to on one slide
subplot(221), plot(t,x(:,1)), axis tight
title('Area vs Time')
xlabel('Time')
ylabel('Area')
subplot(222), plot(t,x(:,3)), axis tight
title('Volume vs Time')
xlabel('Time')
ylabel('Volume')
subplot(223), plot(t,x(:,2)), axis tight
title('Area Rate vs Time')
xlabel('Time')
ylabel('Area Change Rate')
subplot(224), plot(t,x(:,4)), axis tight
title('Volume Rate vs Time')
xlabel('Time')
ylabel('Vol Rate Change')

% Plot A' vs V' (not used in paper)
figure
plot(x(:,4),x(:,2),'g')
title('Rate of Area Change vs Rate of Volume Change')
xlabel('Rate of Volume Change')
ylabel('Rate of Area Change')

function xprime=glacier12(t,x)
g=0.024;
T_A=8.0;
T_V=48;
B=0;
H=123;
A_o=0.094*10^6;
b=5.5;
B_p=0;

xprime=zeros(2,1);
xprime(1)=x(2);
xprime(2)=-(1/T_A-g)*x(2)-(1/(T_A*T_V))*x(1)+B/(T_A*H)+(g/T_A)*A_o;
xprime(3)=x(4);
xprime(4)=-(1/T_A-g)*x(4)-(1/(T_A*T_V))*x(3)+B_p+(B/T_A)+(b/T_A)*A_o;
References


