Exploring a Two Person Zero-Sum Game

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December 18, 2009

Abstract

Many game theory textbooks fail to bridge a gap between simplistic and complicated. Understanding the vector picture of $2 \times 2$ payoff matrices and how they combine linearly with a probability vector can help students visualize certain properties of optimal and dominant strategies. We assume the reader has basic understanding linear algebra. We examine the vocabulary of zero-sum games and then proceed to show methods to solve probability vectors in the players’ mixed-strategy set.

1 Introduction

What drives our decisions? Why don’t we always make decisions in the best interest of ourselves or others? Can we effectively bargain with others by providing certain information? These questions have driven mathematicians to analyze the decision process from a logical and statistical point of view. When I began to think about what kind of human interactions would be of the greatest interest to people, I began to think of the relationship between men and women. People in relationships make big decisions such as marriage, having kids, and unfortunately divorce. But they also make lesser decisions such as whether you should go Dutch on a date, whether (or when) it is okay to ask a date for a nightcap, what kind of present should you get for a special occasion. But even with every one of these decisions the number of variables to consider is too great to even make a simplified version.

Then I thought about politics. It always seems to be on people’s minds these days. In many textbooks about game theory, they include an exercise about the cold war. In fact, when game theory was first being formulated, the government was interested in its discoveries (at the Rand Corporation). Unfortunately, it was the unexamined logic of game theory that justified the weapons race, and the madman theory of foreign policy during the cold war. Now, we are much wiser, and understand that human interaction is complicated. Part of the problem with any mathematical model of human interaction is that there is a wealth of information that is inaccessible, and motivations must be based on fungible utility scales. But how can you compare the joy of having a kid with a monetary (or numerical) value? This is a problem the economists have. They know that any model they have about how the economy works is only at best
an estimation. Better models will yield better estimations. Yet we should take any economic theory with a grain of salt, because assumptions in the logic may not play out in future circumstances.

I also decided that the best introduction to thinking about game theory as vectors and matrices was with the zero-sum game. Most games are not zero-sum. In fact, it’s beneficial when they aren’t because then we can have cooperation. But our examination of the payoff matrix wouldn’t have been possible with any other game.

With that said, I chose to scale back my ambitions and use game theory to describe a simple child’s game. But even in this, there will be some hand-waving.

1.1 Example: Morra

Let’s consider a game of Morra between two players, let’s call them Simon and Garfunkel. The rules of Morra are simple. Each player may choose one of two strategies:

1. play one finger
2. play two fingers

Now, if the sum of the two players’ fingers is even (2 or 4 fingers), Simon wins. If odd (3 fingers), Garfunkel wins. At the end of each round, the loser gives the winner some payoff, let’s say, oh, 10 jelly beans. This isn’t a very interesting game from a strategic point of view because it soon becomes apparent that one strategy is no better than the other. The best one player could do is to avoid a pattern and completely randomize their strategy. So if Simon and Garfunkel are rational players, both should have roughly the same amount of jelly beans at the end of the day.

How about making this game a little morra interesting? Let’s arbitrarily make the rule that if the sum of the two players’ fingers is 4, Simon gets twenty jelly beans from Garfunkel’s stash. If this is the case, then what strategy should Simon play? Simon notices that he can only win twenty jelly beans if he plays two fingers. But he knows that Garfunkel knows that also, so Garfunkel may only play one finger to avoid paying off twenty jelly beans. So maybe Simon should play one finger some of the time. But is there any way for him to know how often he should play one strategy over the other.

In order to better understand this problem, let’s analyze the problem in the language of game theory.

1.2 Background: Vocabulary and Terms

We will briefly give some vocabulary of game theory. Every term in this section is relevant, but complete understanding of the formal abstractions are not necessary for our purposes.

The game described above is known as a two-person zero-sum game with a finite pure strategy set for each player. Each player has a strategy profile or set, \( S \). In our Morra game we assign Simon and Garfunkel their own strategy
sets, \( S_S \) and \( S_G \) with strategies, play one finger, \( s_1 \) or \( g_1 \), or play two fingers, \( s_2 \) or \( g_2 \) such that
\[
\begin{align*}
S_S &= \{ s_1, s_2 \} \\
S_G &= \{ g_1, g_2 \}
\end{align*}
\]

The strategy profile will describe the total strategy available to the players throughout the game. Each element in our strategy set is what is called a pure strategy. A pure strategy is when a player’s decision depends entirely on the situation in the game, and does not arbitrarily change given the same situation. For example, if Garfunkel knew that Simon was always going to play two fingers, the pure strategy would be to play one finger. A pure strategy gives us a complete description of all the possible moves a player may take given any situation in the game. We’ll explain later why this isn’t a pure strategy game.

Because the game is simultaneous, this reduces the number of strategies in the game. When a game is not simultaneous, a player may choose to do the same thing as the other player, or the opposite. This would increase the number of options by two.

Whenever Simon wins some number of jelly beans, Garfunkel loses the same number. So if we add up the total number of jelly beans at the beginning of the game we will find the same number of jelly beans at the end of the game. Zero jelly beans were added, and zero were subtracted. This kind of game is called a zero-sum game.

In the Morra game above, jelly beans are what is called the payoff of the game. Whenever Simon and Garfunkel play a round of Morra, there is a certain payoff associated with it. We will call this mapping of two strategies to a payoff the payoff function,
\[
J(s_i, g_j)
\]

where \( s_i \) and \( g_j \) are in Simon and Garfunkel’s strategy set. Notice that we are only able to assign one payoff for both players because it is a zero-sum game. In other words, when Simon wins 10 jelly beans, then Garfunkel loses 10 jelly beans, and the payoff function \( J(x, y) = 10 \). On the other hand, when Garfunkel wins 10 jelly beans, \( J(x, y) = -10 \). So by the end of the game, Simon wants a positive payoff and Garfunkel wants a negative payoff.

If there is a unique strategy in Simon’s strategy set where given any strategy Garfunkel adopts Simon’s payoff is maximized, we say that this strategy strictly dominates all other strategies in the set.
\[
\begin{align*}
&\text{If } \exists s^o \in S_s \ni J(s^o, g_j) > J(s_i, g_j) \\
&\forall s \in S_s, \text{ then } s \text{ strictly dominates.}
\end{align*}
\]

With a dominant strategy, the player will always play this pure strategy in the game.

However, in the Morra game no single strategy strictly dominates any another. If a player only played one strategy, the other would adapt and play the
strategy that would defeat it. In this case, the best strategy is a mixture of strategies, playing each strategy some of the time. In a mixed-strategy game, we assign each pure strategy element in each strategic set a corresponding probability element. We’ll use $x_i \in X$ for the probability elements associated with Simon’s strategic set and $y_j \in Y$ for Garfunkel’s. Each element in the probability sets have the property that

$$\sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad x_i \geq 0$$

$$\sum_{j=1}^{n} y_j = 1 \quad \text{and} \quad y_j \geq 0$$

If the Morra game was fair (Simon gets 10 instead of 20 jelly beans), the two strategies in both players’ strategy sets (either to hold up one finger or two) would be considered intransitive. This means a player would do no better playing one strategy more often than the other. In the fair case, the best mixed-strategy would be to play each finger exactly 50 percent of the time. Games with intransitive strategies such as a fair game of Morra or rock-paper-scissors have equal probabilities for each strategy. In other words, if there are $n$ elements in our strategy set, then the probability associated with each element is $1/n$. But we should make clear that a fair game doesn’t necessarily mean that the strategy sets will be intransitive. A fair game simply means that given optimal strategies from all parties no one has an advantage to win more than the other.

The assumption that a player will execute the best strategy when given all other information about the game should be addressed. We assume that Simon and Garfunkel have the same access to information in the game and are rational. This generally means Simon and Garfunkel use strategic means to maximize utility (the number of jelly beans they each get). We also allow Simon and Garfunkel access to each other’s strategies. This affords us having to adapt our model to include the possibility of one player acting willy-nilly, which would complicate our analysis of the payoff after a certain number of games.

The fundamental problem of our Morra game is to find a solution whether there is a mixed-strategy that Simon can take that doesn’t offer any advantage to Garfunkel and vice versa. If there is such an equilibrium, we can make a prediction of a payoff given these two mixed-strategies. This equilibrium point is known as a saddle point for the mixed-strategy. It is the maximum payoff Simon is assured when Garfunkel chooses a minimizing strategy. This payoff is called the minimax because it minimizes a maximum. We will denote each strategic set that accomplishes this by the superscript ’o’ for optimal or

$$J(X^o, Y) \geq J(X, Y) \quad \forall X$$

$$J(X, Y^o), \leq J(X, Y) \quad \forall Y$$

### 1.3 Modeling the Game with Matrices

The zero-sum game’s payoff function can be represented by a $m \times n$ matrix, where $m$ is the number of strategies in the strategic profile of player $y$ and $n$ is the number of strategies in the strategic profile of player $x$. Then the component
$A_{ij}$ corresponds to the strategy $i$ for player $y$ and $j$ for player $x$. Examples of payoff matrices:

$$
\begin{pmatrix}
1 & 3 & 8 & 9 \\
0 & 2 & 0 & 3 \\
1 & 6 & 4 & 4 \\
\end{pmatrix}
$$

The first example has a pure strategy solution. This can be found by eliminating dominated strategies. We do this by looking at either the columns (for player $x$) or the rows (for player $y$). Player $y$ wants to minimize the score. Comparing the rows, strategy 2 is strictly better because whatever column that is chosen, strategy two will lead to a lesser value. Player $x$ wants to find the maximum, but only in row two, because player $x$ knows that player $y$ will choose the minimized strategy. Therefore, the best strategy for player $x$ is in column 4. So the pure strategy saddle point is in the corner at $A_{24}$.

The second example does not have a pure strategy solution. It’s optimal solution is a mixed-strategy. This example actually has more than one optimal mixed-strategy for player $y$ in its strategic set. In this example, row two is dominated by the other three strategies and will not enter into the optimal solution. We’ll discuss how to find an optimal solution in the next section.

Our payoff matrix for the Morra game described above is

$$
A = \begin{pmatrix}
10 & -10 \\
-10 & 20 \\
\end{pmatrix}
$$

where player $x$ is Simon and player $y$ is Garfunkel. The columns represent the two strategies for Simon and the rows represent the two choices for Garfunkel. For example, if both Simon and Garfunkel play 1 finger, the payoff is 10 jelly beans for Simon. This corresponds to the position $a_{11} \in A$ such that the payoff $J(1, 1) = 10$. Similarly, if Simon plays one finger and Garfunkel plays two, the payoff is $a_{12} = J(1, 2) = -10$, or 10 jelly beans for Garfunkel. We can now see how to apply a randomized strategy to the payoff matrix.

First, let’s look at what options Simon has and then later on bring Garfunkel into it. Let’s consider the unknown probability set $X$ of Simon as a two-dimensional vector, $x$ with conditions $\sum_{i=1}^{n} x_i = 1$ which will maximize the payoff. The two payoffs possible to Simon independent of Garfunkel’s strategy are the components of result of the payoff matrix multiplied by the probability set vector, $Ax$. This game does not contain any dominated strategies and so when Simon maximizes the components of the resultant $Ax$ components will be equivalent. Because if one component were greater than the other, Garfunkel would find the strategy that would choose the lesser of the two for Simon. And he would do better to use a probability strategy that would result in a vector with equal components. Therefore, if there exists a optimal probability set of
strategies, $X^0$, the payoff matrix $A$ will map it onto the line, $y = x$. Simon wants this vector’s magnitude as large as possible. In other words, what linear combination of the column vectors will give Simon the greatest payoff?

$$A\mathbf{x} = \begin{pmatrix} 10 & -10 \\ -10 & 20 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 10 \\ -10 \end{pmatrix} + x_2 \begin{pmatrix} -10 \\ 20 \end{pmatrix}$$

Notice that the column vectors of our payoff matrix when plotted on an $xy$-axis are on opposite sides of the line $y = x$ (please refer to the graph below). This tells us that the optimal linear combination will have a mixed-strategy, because if they were on the same side, there would be a pure strategy that strictly dominates. The pure strategy would be found by choosing the vector whose head is furthest to right if both vector are above the line $y=x$, and the vector whose head is furthest to the left if both are below the line $y=x$. A pure strategy might also exist if there is a vector whose least value of either $x$ or $y$ coordinate is greater than the linear combination vector on the line $y=x$ (please refer to the Pure Strategy graph). We can expand this idea of pure strategy to multidimensions where there might be a combination of vectors less than the number of basis vectors in our space that have a greater value for every component than the best linear combination to the ones vector (the vector whose components leads to a scalar multiple of the vector with all 1s for components).

The graph also shows us that the mixed-strategy is not intransitive, because the two column vectors are not reflected across the line $y = x$.

If we have an underdetermined or overdetermined payoff matrix, we will most likely be able to eliminate strategies until we arrive at a square payoff matrix. We can better see this with the vector picture. We will come back to this in the next section.

It is easy to see an unfair game when there lacks symmetry in the payoff matrix and graphical representations. However, not every symmetric payoff matrix will represent a fair game. Consider a game where the payoff matrix is the $n \times n$ identity matrix. This could be like a game of battleship where player $y$’s ship lies on the diagonal of a $n \times n$ grid. Player $x$ and $y$ both choose a number less than or equal to $n$. If that number is the same, $y$ loses a point. Although, the game is intransitive, where both players will adopt the mixed strategy of $1 \div n$, the payoff certainly won’t be zero. (We will leave it to the reader to later determine that the average statistical payoff is also $1 \div n$).

We can also check to see if a game is fair by looking at the nullspace of the payoff matrix. A fair game means that the payoff must be zero. And if there does not exist a strictly dominating strategy (ie the optimal strategy, $X^0$, gives $A\mathbf{x}$ where all components are equal), the nullspace must include at least one nonzero vector whose components add up to one. That is another reason why the identity matrix as a payoff does not represent a fair game.

There is one type of payoff matrix that will always result in a fair zero-sum game. Remember that when Simon wins a number of jelly beans, Garfunkel
Vector Picture of Payoff Matrix

$y = x$

(-10, 20)

(10, -10)

Line of possible linear combinations
where sum of coefficients equals 1

Payoff vector (2, 2)
Vector Picture of Payoff Matrix w/ Pure Strategy Solution

Player x: Maximizing

False Maximized Security Point

True Maximized Security Point
loses the exact number. We could write Payoff to Simon $=$ - Payoff to Garfunkel. Also, while Simon’s payoff vector is in the column space of the payoff matrix, Garfunkel’s is in the row space. So $A^T y$ will give Garfunkel’s payoff vector. So $-A^T = A$ will give us payoffs of zero. Therefore, the skew-symmetric payoff matrix will give us a fair zero-sum game.

Examples of payoff matrices of fair games:

\[
\begin{pmatrix}
10 & -10 \\
-10 & 10
\end{pmatrix}
\]

(a) Symmetric w/ Nonzero Nullspace and Sum of Coefficients $=$ 1

\[
\begin{pmatrix}
0 & -1 & -1 \\
1 & 0 & -1 \\
1 & 1 & 0
\end{pmatrix}
\]

(b) Skew-symmetric

In the skew-symmetric example above, the game isn’t intransitive. By eliminating dominated strategies, we find that the pure strategy that corresponds to $A(1, 1)$ will dominate all other strategies.

We now look at Garfunkel’s strategy. Given any mixed strategy Simon employs, or any vector $x$, Garfunkel wants to minimize the payoff. The payoffs available to Garfunkel in the payoff matrix are in the row space. Therefore, the vector $y$ that have components in the probability set $Y$ can be thought of as multiplied by the transpose of the payoff matrix, $A^T y$.

By graphing the row vectors in $A$, we get the exact same picture as the column picture. This is because our payoff matrix is symmetric, $A^T = A$. The only difference is that Garfunkel wants a linear combination of the rows that minimize the vector on the line $y = x$. An important property of a symmetric payoff matrix is that the mixed strategy for both players as well as the payoff solution will be identical. The probability vector in the probability set of one player will be identical to the probability vector in the other.

If there exist solutions to $x$ and $y$, then we can solve for the payoff,

$$\sum_{i=1}^{2} J(x_i s_i, y_j g_j) = y^T A x = \text{Statistical Average Payoff}$$

Dimensional analysis of the matrices will show that this multiplication will result in a number. That number is the total payoff for the game.

This payoff isn’t certain for any number of games. It is only the most likely average outcome when the game is repeated. When we look at the solution for our Morra game, we’ll see that the solution for our payoff actually can never be reached no matter how many game are played. The solution represents average gain or loss of the player over a number of games. We’ll examine this further when solve the minimax for the payoff matrix.
When one of the two player has only two or three choices, we should be able to estimate what our payoff solution will be. First consider our Morra game. We'll look at the vector picture for our payoff matrix columns. Simon (player \( x \)) wants to find a linear combination of the column vectors that will secure a maximum payoff. Garfunkel wants to find a linear combination of the row vectors that will minimize Simon’s payoff. Our payoff matrix is symmetric, so our row vectors are the same as our column vectors. So we can look at both pictures at the same time.

\[
\begin{align*}
\min_y y^T A x & = \max_x \min_y y^T A x \\
\max_x y^T A x & = \min_y \max_x y^T A x
\end{align*}
\]

### 1.4 Finding a Solution using the Vector Picture

It may have occurred to you that finding a solution for the \( 2 \times 2 \) example only requires a bit of algebra. We'll let the cat out of the bag after we first look at how to solve it using a little bit of linear algebra. This is because when the game becomes larger than a \( 2 \times 2 \) example, or if we have a rectangular matrix, we can’t use simple algebraic manipulation to find our solution.

Remember the vector picture of our Morra game. We had the same two vectors (row vectors and column vectors) because matrix \( A \) is symmetric. The coefficients on the linear combinations of the two vectors must add up to one and equal some scalar of the vector \( (1,1) \). The coefficients must add up to one because of the probability constraint. This means that the resultant vector must lie on the line that connects the ends of the two vector heads. We can further constrain our resultant vector to lie also on the line \( y=x \) (ie be some scalar of \( (1,1) \)) because any other resultant vector has a component that is strictly less than either component of this scalar of the \( (1,1) \) vector. Remember that if player \( x \) chooses a strategy that leaves player \( y \) the ability to find a solution that chooses the smaller of the two components, player \( y \) will minimize this. So player \( x \) wants to find a maximum security so that no matter what \( y \) does, player \( x \) is assured of this payoff. We can think of this graphically by finding the line between our two vector heads and finding the intersection of this line with the line \( y=x \). Now, if we move along the line between our two vectors and below the line \( y=x \), then we may have a larger \( x_1 \) component, but we also have a lesser \( x_2 \) component. So if player \( y \) is rational (and we assume he has knowledge of what player \( x \) is going to do), player \( y \) will find the strategy that selects this lesser \( x_2 \) component. In other words, if player \( y \) could encounter the resultant column vector \( A x \), player \( y \) would either select the strategy \( (1,0) \) or \( (0,1) \) to minimize \( y^T A x \).

So we now have one way we can solve this problem. We can find the line between our vector heads and intersect it with \( y=x \). This will give us the payoff vector \( A x \) (which is also \( A y \) because of symmetry). Now, all we need to do is solve the familiar \( A x = b \) where \( b \) is the payoff vector. Solving for our line between the two vectors \( (10,-10) \) and \( (-10,20) \), we find the line \( y = -1.5x + 5.5 \). The intersection of this line with the line \( y = x \) gives us our payoff vector...
(2,2). (Remember, that in our game, never will there be a time when Simon or Garfunkel will actually have 2 jelly beans. How could they when they only deal with winning/losing 10 or 20 a round? This number is the statistical average of jelly beans won or lost each game.)

Performing rref on the augmented matrix $A\mathbf{b}$ gives us our solution for $x$, or $x_1 = .6$ and $x_2 = .4$.

$$A\mathbf{x} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} .6 \\ .4 \end{pmatrix}$$

This is also the solution for $y$. (But it is important to note again that this is only so because of the symmetry in our game).

This solution might at first seem counterintuitive. Even though Simon will receive a payoff of 20 every time he wins with two fingers, the percentage of the time he plays this strategy is only 40 percent. This is the best strategy given our assumptions about Garfunkel maximizing his payoff.

While we’re looking at the vector picture, let’s momentarily look at how an underdetermined payoff matrix can be reduced to a square matrix. (Any overdetermined matrix can be transposed to an underdetermined, so our example will cover both). Consider the payoff matrix below.

$$\begin{pmatrix} 5 & -1 & 4 & 3 & -8 \\ 0 & 3 & 8 & -5 & 7 \end{pmatrix}$$

(a) an underdetermined payoff matrix

We can plot the column vectors on a cartesian coordinate axis. Now, we connect the ends of the vectors that cross the line $y = x$. Player x will want to choose the combination that produces the largest scalar multiple vector of (1,1). We find that the combination of the first and third column vector produces this maximization. Therefore, the mixed strategy will only consider strategy 1 and strategy 3. So we can “throw away” the other three vectors and produce a square matrix. Then we can explore both the column space and row space without having to deal with the 5-dimensional vectors in the original row space. We can also perform rref to find our solution. But here is an example when our matrix isn’t symmetric. So we will have two different probability vectors as our solutions for player x and player y. (We provide vector pictures for both players below). We will leave it to the reader to determine that the payoff vector is $(40/9,40/9)$ and the solution is $x = (4/9,5/9)$ (remember that $x_2, x_4, x_5$ are equal to zero) and $y = (8/9,1/9)$.

This idea of finding the linear combination of vectors with coefficients whose sum equals one that produces the largest scalar multiple of the ones vector (ie the vector whose components all equal one) is one of the best way to think about mixed-strategies visually. You can apply this method to multidimensions. Even though it may be difficult to visualize n-space when n is greater than 3,
Vector Picture of Underdetermined Payoff Matrix

Player x: Maximizing

Maximized Security Point

$(40/9, 40/9)$

$(4, 8)$

$(5, 0)$
Vector Picture of Underdetermined Payoff Matrix

Player y: Minimizing

Minimized x’s max

(40/9, 40/9)

(0, 8)

(5, 4)
be analogy we can "see" what’s going on when we reduce the payoff matrix by eliminating dominated strategies or find where the heads of our vectors intersect the ones vector. We can also "see" when the mixed-strategy has a solution that doesn’t exactly map into the ones vector because of the existence of strictly dominated strategies or having the significant vectors all above or below the line $y = x$.

So now we will let the cat out of the bag. We’ll solve our game the old fashioned way. If our payoff matrix is a scalar multiple of the ones vector (which in our case it is, but you have to eliminate the possibility of dominating strategies first), then we can solve the two equations by setting them equal to each other and making the substitution $x_1 = 1 - x_2$. Then,

$$10x_1 - 10(1 - x_1) = -10x_1 + 20(1 - x_1)$$

Solving for $x_1$ and then back substitution for $x_2$, we find the same solutions $x = y = (.6, .4)$. But this isn’t as satisfying or useful as getting the vector picture because once we leave our comfortable 2 or 3 dimensional space, where we are dealing with multidimensional vectors, and computations will become nearly impossible by this brute force method.

### 1.5 Finding a Solution by Linear Programming

There is another way to solve the strategic vectors in our game. It involves a bit of linear programming. If we had more time, we could show how to solve any solvable game with any number of strategies by transforming our game into a linear programming problem. In fact, if we were serious about learning game theory, we would want to learn how solve linear programming problems. However, for our purposes, we will simply show a rough sketch on how it works.

**Linear programming** is a part of optimization theory. It differs slightly from what we have studied in linear algebra because our regular linear equalities now become linear inequalities under constraints. Every linear programming problem has a **cost or objective function** (it’s called cost because it is usually dealing with minimizing the cost and maximizing profits). Our cost function is simply a horizontal line, $y = k$, such that as we go up the graph the higher the value of the objective function. So when our objective function is at 10, that is payoff solution for our game.

The constraints for player x in our game come from the lines that we saw above,

$$10x_1 - 10(1 - x_1)$$

and

$$-10x_1 + 20(1 - x_1)$$
We want to set these inequalities so that they are all greater than or equal to zero. So we will add 10 to each component of our payoff matrix. This will not effect our solution for $x$, but we will need to remember to subtract 10 if we want to see the payoff.

Now if we plot the inequalities on the $xy$-plane, we can find the basic feasible set. In linear programming, everything in the basic feasible set can be reached. But with game theory, we are only concerned with the lines at the edge of the basic feasible set. Either way, solutions to linear programming problems will always exist at corners or pivots anyway. So we shouldn’t worry about this difference. See graph below.

Linear programming maximizes by taking the objective function and moving it to higher values until it reaches the pinnacle of the basic feasible set. It minimizes by going in the opposite direction. One interesting feature of the minimizing and maximizing problem is that they are the same in different contexts. We’ve already shown that player x’s maximizing of player y’s minimizing solution is exactly the same as player y’s minimizing of player x’s maximizing solution. This property of linear programming problems is known as duality.

One way to solve a linear programming problem is known as the Simplex Method. This method finds a starting corner and finds the next corner that gives a higher value for our objective function (or lower if we are minimizing). This seems like a trivial task that one could do by eyeing it. With our example, we can find all the the corners and find the solution faster than we could write a program to solve it for us. But when the number of constraints becomes too great, and it becomes impossible to visualize all the intersecting $n$-dimensional planes, we need a method to find the corner for us. We encourage the reader to read more about the Simplex Method and a newer alternative method called Karmarkar’s Method. While the Simplex Method starts at the outside of the basic feasible region, Karmarkar’s method starts at the inside and uses projections to find the optimal solution.

Looking at our Morra game through a linear programming lens, we see the same solution that we did when we looked at the vector picture.

Now let’s see what happens to the strategic solutions of our Morra game if we were to underdetermine the matrix. Let’s give Garfunkel two extra options to help make this game fairer,

$(3)$ play 4 fingers $(4)$ play 6 fingers

We will also change the payoff matrix so that Simon wins 0 jelly beans when the sum of the fingers is 5 and 15 jelly beans when the sum is 6, and loses 3 jelly beans when the sum is 7 and wins 5 jelly beans when the sum is 8. So our new payoff matrix is,

$$
\begin{pmatrix}
10 & -10 & 0 & -3 \\
-10 & 20 & 15 & 5
\end{pmatrix}
$$

The payoffs for these two options seem to favor Garfunkel. He has more choices, and he never has to lose 20 jelly beans because he can choose either
Finding Basic Feasible Region

Player y choosing 2 fingers
Line: $-30x + 30$

Player y choosing 1 finger
Line: $20x$

Player x's security

Pivots = 16

Basic Feasible Region
for player x: Inside triangle

Probability of x choosing one finger
Payoff

Probability of x choosing one finger
Payoff
Finding Basic Feasible Region

Player y choosing 1 finger
Line: 20y

Player y choosing 2 fingers
Line: -30y + 30

Feasible Region for player y: Inside pentagon

Player y's minimization
option 3 or 4. In fact, if we look at the vector and linear programming picture we find that the game does become a lot fairer. Player 1 will play 1 finger less, only 54 percent of the time (as opposed to 60) and the statistical average payoff will be only .715, just barely a better advantage for Simon. See graph at end of paper.

1.6 Concluding Remarks

Although I began researching game theory hoping to do something with adaptive learning, I became interested in simpler ways to address zero-sum games. And while the textbooks I read implicitly allude to the vector model, it rarely is spelled out. I wanted something for a simple mind.

Finally, I would just like to make a few remarks about what the games we looked at were and were not. Our assumptions about the players’ being able to have perfect recall and knowledge about the other player's strategy makes our model of the game rather unrealistic. People are not always rational beings (who are known as homo economicus). Modern attempts at game theory attempts to address the problem of rationality when offering analysis (ie bounded rationality).

Another assumption we make that the number of jelly beans either player has at any time doesn’t change their strategy. In economics, this is known as diminishing marginal utility, or each additional dollar is of less utility to person than the previous dollar. We also will not consider the complementary concept of risk aversion, or how likely a player is willing to make a risky bet for a higher payoff.

Game theory offers some good logical enjoyment. It has been applied to every social science and even found a place in evolutionary science, where the gene-environment field is the player and the game is played on all time scales, from seconds to millions of years. If I were to delve deeper into game theory, I would choose to enter here.

References


Probability of $x$ choosing one finger

Payoff

Underdetermined Basic Feasible Region

Lines:

- $-30x + 30$
- $20x$
- $-10x + 25$
- $-8x + 15$

Basic Feasible Region

for player $x$: Inside rhombus